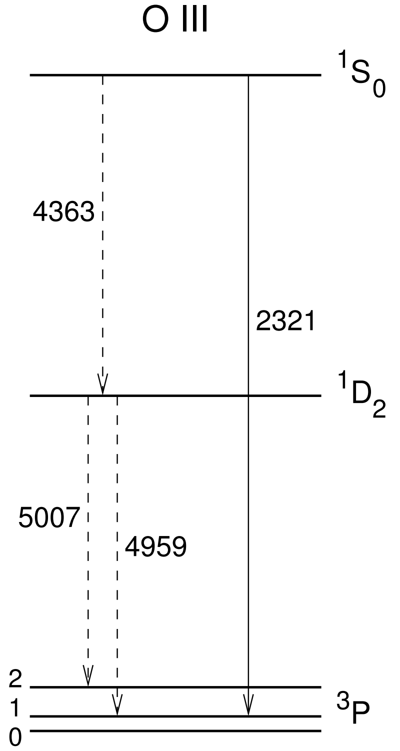
Question 1: *Temperature Diagnostics*



In this exercise we will be looking at the atomic physics behind density diagnostics, using the [O III] multiplets as an example. The relevant term scheme is shown at the left. In the following we will call the ^{3}P ground state level 1, while ^{1}D$_2$ is level 2 and ^{1}S$_0$ is level 3.

a) In plasmas of sufficiently low density, higher levels are excited via collisions but collisional de-excitation can usually be ignored, i.e., for two levels 1, and 2, we can assume $n_\mathrm{e}C_{21} \ll A_{21}$. Using this information write down the rate equations connecting levels 1, 2, and 3 and solve them for the density ratio n_3/n_2. You may assume that $C_{23} = 0$.

Solution: With the information given above

$$n_2 A_{21} = n_1 n_\mathrm{e} C_{12} \tag{s1.1}$$
$$n_3(A_{31} + A_{32}) = n_1 n_\mathrm{e} C_{13} \tag{s1.2}$$

giving

$$\frac{n_3}{n_2} = \frac{C_{13}}{C_{12}} \frac{A_{21}}{A_{31} + A_{32}} \tag{s1.3}$$

b) For an optically thin plasma, the relation between the line intensity of a line at frequency ν_ul and the Einstein-A-coefficient is

$$4\pi I_\nu = n_\mathrm{u} \nu_\mathrm{ul} A_\mathrm{ul} \tag{1.1}$$

where u denotes the upper and l denotes the lower level. Derive an equation for the intensity ratio of the 4636 Å and the sum of the 5007 Å and the 4959 Å lines (the latter shall be described by A_{21}), i.e., for $I(4363\,\text{Å})/(I(5007\,\text{Å}) + I(4959\,\text{Å}))$.

Solution: With the information given and Eq. (s1.3)

$$\frac{I(4363\,\text{Å})}{I(5007\,\text{Å}) + I(4959\,\text{Å})} = \frac{n_3 \nu_{32} A_{32}}{n_2 \nu_{21} A_{21}} = \frac{C_{13}}{C_{12}} \frac{A_{21}}{A_{31} + A_{32}} \frac{\nu_{32}}{\nu_{21}} \frac{A_{32}}{A_{21}} = \frac{C_{13}}{C_{12}} \frac{A_{32}}{A_{31} + A_{32}} \frac{\nu_{32}}{\nu_{21}} \tag{s1.4}$$

c) The collisional excitation rate, C_{ji}, is given by

$$C_{ji}(T) = \int_{v_{\min}}^{\infty} v \sigma_{ji} f(v)\, dv \tag{1.2}$$

where the minimum excitation velocity is

$$v_{\min} = \sqrt{2E_{ij}/m_\text{e}} \tag{1.3}$$

and where the cross section is given in terms the collision strength, Ω_{ji}, as

$$\sigma_{ji}(v) = \left(\frac{\pi h^2}{4\pi^2 m_\text{e}^2 v^2}\right) \frac{\Omega_{ji}}{g_j} \tag{1.4}$$

i) For a thermal plasma with a Maxwellian velocity distribution

$$f(v) = \sqrt{\frac{2}{\pi}\left(\frac{m_\text{e}}{kT}\right)^3} \cdot v^2 \cdot \exp\left(-\frac{mv^2}{2kT}\right) \tag{1.5}$$

show that

$$C_{ji} = \sqrt{\frac{h^4}{8\pi^3 m_\text{e}^3 k}} \cdot \frac{\Omega_{ji}}{g_j} \cdot T^{-1/2} \cdot \exp\left(-\frac{E_{ij}}{kT}\right) \tag{1.6}$$

- It will be useful to know that

$$\int x \exp(-kx^2)\, dx = -\frac{1}{2k} \exp(-kx^2) \tag{1.7}$$

- Inserting all constants, the square-root evaluates to $8.629 \times 10^{-6}\,\text{cm}^3\,\text{s}^{-1}$.

Solution: Inserting the equations yields

$$C_{ji} = \frac{\pi h^2}{4\pi^2 m_\text{e}^2} \frac{\Omega_{ji}}{g_j} \sqrt{\frac{2}{\pi}\left(\frac{m_\text{e}}{kT}\right)^3} \int_{\sqrt{2E_{ij}/m_\text{e}}}^{\infty} v \exp\left(-\frac{m_\text{e} v^2}{2kT}\right) dv \tag{s1.5}$$

which after integration results in

$$= \frac{\pi h^2}{4\pi^2 m_\text{e}^2} \frac{\Omega_{ji}}{g_j} \sqrt{\frac{2}{\pi}\left(\frac{m_\text{e}}{kT}\right)^3} \cdot \frac{kT}{m_\text{e}} \cdot \exp\left(-\frac{m_\text{e}}{2kT} \frac{2E}{m_\text{e}}\right) \tag{s1.6}$$

and collecting all terms then gives Eq. (1.6).
For somebody calculating the square-root numerically, it is best to do this calculation using logarithms since one runs into floating-point underflows otherwise.

ii) Show that $I(4363\,\text{Å})/(I(5007\,\text{Å}) + I(4959\,\text{Å}))$ depends only on temperature and on atomic constants.

Solution: Inserting Eq. (1.6) into Eq. (s1.4) gives

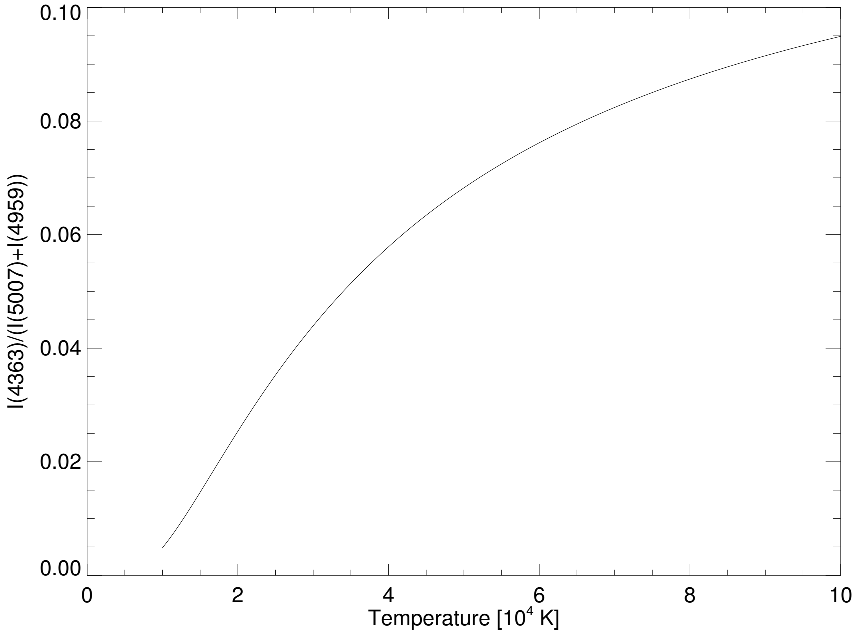
$$\frac{I(4363\,\text{Å})}{I(5007\,\text{Å}) + I(4959\,\text{Å})} = \frac{\Omega_{13}}{\Omega_{12}} e^{-E_{32}/kT} \left(\frac{\nu_{32}}{\nu_{21}}\right) \frac{A_{32}}{A_{31} + A_{32}} \tag{s1.7}$$

d) *(To be done at home with a computer)* For [O III], the relevant atomic constants are $E_{32} = 2.84\,\text{eV}$, $A_{32} = 1.8\,\text{s}^{-1}$, $A_{31} = 0.221\,\text{s}^{-1}$, $\Omega_{13} = 0.28$, and $\Omega_{12} = 2.17$. Plot the line ratio.

Solution: We have

$$\frac{I(4363\,\text{Å})}{I(5007\,\text{Å}) + I(4959\,\text{Å})} = 0.132 \times e^{-32990/T} \tag{s1.8}$$

and this gives the following plot



(yes, this looks different than the figure in the lecture notes, which shows the inverse of the above, i.e., $(I(5007\,\text{Å}) + I(4959\,\text{Å}))/I(4363\,\text{Å})$).

Note: that for densities around the critical density, de-excitation cannot be ignored, which will result in more complicated equations (see lecture notes). For [O III] the critical density of the $^1D_2$–$^3P_2$-transition is $8.6 \times 10^4\,\text{cm}^{-3}$.