Astronomy from Space
Example Exam Question

Question 1: *Neutron Star Accretion*

(a) Typical magnetic fields of neutron stars are $B = 10^7\,\mathrm{T}$. Because of this strong magnetic field, matter accreting on a neutron star will not form an accretion disk, but fall along the magnetic field lines onto the magnetic poles of the neutron star.

(i) Derive a formula for the velocity of matter accreting from infinity onto the surface of a neutron star with mass M and radius R. You may assume free fall. Ignore relativistic effects.

Solution: The free fall velocity can be found from energy conservation arguments, i.e., the sum of gravitational potential energy and kinetic energy is

$$-G\frac{Mm}{R} + \frac{1}{2}mv^2 = 0 \tag{s1.1}$$

since the potential energy at infinity is 0 and we assume the accreted material is a rest at infinity. Therefore, the free fall velocity is

$$v = \sqrt{\frac{2GM}{r}} \tag{s1.2}$$

(ii) Using your result from subquestion ai, calculate the free fall velocity for a $1.4\,M_\odot$ neutron star with 10 km radius and express it $\mathrm{m\,s^{-1}}$ and as a fraction of the speed of light.

Solution: Inserting numbers into Eq. s1.2 gives $v = 1.9 \times 10^8\,\mathrm{m\,s^{-1}} = 0.64c$

(iii) At the surface of the neutron star, the accreted material comes to an abrupt halt. Due to energy conservation, the kinetic energy of the accreted material is converted into thermal motion. By setting the kinetic energy of the accreted particles equal to $\frac{3}{2}kT$, determine the temperature of the accreted gas at the magnetic pole of the $1.4\,M_\odot$ neutron star and give it in both, Kelvins and keV. You may assume that the accreted gas is pure hydrogen. Note that because of the short timescales involved, electrons and protons will not be in thermodynamic equilibrium, i.e., they will have different temperatures.

Solution: The temperature of particles with mass m is

$$T = \frac{m_\mathrm{p}v^2}{3k} \tag{s1.3}$$

For protons, we find

$$T_\mathrm{p} = 1.46 \times 10^{12}\,\mathrm{K} \tag{s1.4}$$

Therefore

$$kT_\mathrm{p} = 2 \times 10^{-11}\,\mathrm{J} = 1.3 \times 10^5\,\mathrm{keV} \tag{s1.5}$$

The temperature of the electrons is smaller by a factor $m_\mathrm{p}/m_\mathrm{e} = 1835$, i.e., $T_\mathrm{e} = 8 \times 10^8\,\mathrm{K}$ corresponding to 70.8 keV.

(iv) Due to the deceleration of the accreted material, soft photons will be emitted. Name the formula which describes the emission of radiation due to the deceleration.

Solution: The formula describing the emission of radiation by an accelerated charge is called Larmor's formula.

(v) The radiation resulting from the deceleration described above is called *bremsstrahlung* (breaking radiation). In neutron stars, typical *bremsstrahlung* photon energies are on the order of a few keV. These photons must propagate through the hot plasma formed by the accreting matter. Describe the expected spectral shape expected, including an estimate the maximum photon energy emitted by the accretion column. Explain your reasoning. Why can you ignore the hot protons?

Solution: The soft bremsstrahlung photons will interact with the ~70 keV hot electrons through Compton scattering. Since the photons have a lower energy than the electrons, inverse Compton scattering will upscatter the photons to higher energies. This is possible until the typical photon energy corresponds to that of the electrons, where the photons cannot gain further energy from the electrons. From Monte Carlo simulations shown in the lectures we know that the resulting spectral shape will be a power law spectrum, which is cutoff at a photon energy corresponding to roughly kT_e.
Protons can be ignored since the relevant cross section for the scattering of photons off a charge is the Thomson cross section. Since the Thomson cross section scales as m^{-2}, the probability that photons scatter off protons is smaller by a factor $1835^2 = 3 \times 10^6$.

(b) We now look at the physical processes in the strong magnetic field at the polar cap in greater detail.

(i) A nonrelativistic electron is moving with velocity v perpendicular to a magnetic field of strength B. Why is the resulting orbit a circle and what is its radius?

Solution: Because the electron is moving perpendicular to the magnetic field, the Lorentz force $F = qv \times B$ will be perpendicular to the electron's motion, resulting in circular motion. The radius can be found from balancing the centripetal force and the Lorentz force:

$$\frac{mv^2}{R} = qvB \tag{s1.6}$$

such that the Larmor radius is

$$R = \frac{mv}{qB} \tag{s1.7}$$

(ii) For sufficiently high B-fields, the Larmor radius will be small enough that quantum mechanical effects become important. This is generally the case once the characteristic length of the electron's orbit becomes comparable to the de Broglie wavelength, $\lambda = h/p$, of the electron. In analogy to the Bohr atom, this leads to a quantization of the electron's energy. In quantum mechanics it is shown that the proper quantization condition is that the circumference of the electron's orbit must be an even integer multiple of the de Broglie wavelength. Show that this leads to quantized electron energies which are proportional to B.

$$E_n = n\frac{\hbar q}{m_e}B \tag{1.8}$$

where n is an integer. These energies are called Landau levels.

Solution: Setting R equal to $2n\lambda$, where n is an integer gives

$$\frac{2\pi}{qB}\sqrt{2mE_n} = 2n\frac{h}{\sqrt{2mE_n}} \tag{s1.8}$$

and solving this with respect to E gives

$$E_n = n\frac{q\hbar}{m}B \tag{s1.9}$$

where $\hbar = h/2\pi$

Remark: The proper derivation of the Landau levels requires knowledge of electrodynamics and quantum dynamics which is outside of the level of this course. The question lies in that the proper quantization condition is different from that given in the question. In reality, one quantizes the electron's angular momentum $\vec{p} \times \vec{r}$, where $\vec{p}$ is the electron's generalized momentum, which is given by $\vec{p} = m\vec{v} - \frac{q}{2}Br\hat{\phi}$, where $\hat{\phi}$ is an unit vector along the azimuthal coordinate. Using this correct approach, one does not have to invoke the non-intuitive condition that only even multiples of λ are allowed.

(iii) The continuum photons described in part (a) of this question can scatter off the electrons in the Landau levels. Analoguous to photons interacting with the quantized electrons around a proton in a hydrogen atom, energy can be transferred from the photons to the electrons. This is only possible if the photon has an energy corresponding to the difference between two energy levels of the electron. As the scattering photon looses energy, an absorption line is observed in the spectrum. In the source Hercules X-1, such a line was discovered at a photon energy of 40 keV, corresponding to the energy between the $n = 1$ and the $n = 2$ Landau level. Determnine the magnetic field at the poles of the neutron star in this system.

Solution: The energy difference between the two levels is

$$E_{2\to1} = \frac{\hbar q}{m_{\mathrm{e}}} B \tag{s1.10}$$

Inserting the constants and expressing the energy in keV gives

$$E_{2\to1} = 10^{-7}\,\mathrm{keV\,T^{-1}}\,B \tag{s1.11}$$

Therefore the measured B-field is 4×10^{8} T.

Remark: Such lines are called Cyclotron lines, they are the only direct way we have to directly measure magnetic fields of neutron stars.

The following are constants that might be useful for solving the example questions.

$$\begin{aligned}
h &= 6.63 \times 10^{-34}\,\mathrm{J\,s} \\
c &= 3 \times 10^{8}\,\mathrm{m\,s^{-1}} \\
1\,\mathrm{eV} &= 1.6 \times 10^{-19}\,\mathrm{J} \\
q &= 1.6 \times 10^{-19}\,\mathrm{C} \\
m_{\mathrm{e}} &= 9.1 \times 10^{-31}\,\mathrm{kg} \\
m_{\mathrm{p}} &= 1.67 \times 10^{-27}\,\mathrm{kg} \\
M_{\odot} &= 2 \times 10^{30}\,\mathrm{kg} \\
G &= 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}} \\
k &= 1.38 \times 10^{-23}\,\mathrm{m^2\,kg\,s^{-2}\,K^{-1}}
\end{aligned}$$