

Synchrotron Radiation



Introduction

To obtain the synchrotron radiation spectrum, we will have to perform the following steps:

1. Derive the **motion of electrons in magnetic fields**
2. Then use Larmor's formula to obtain the **radiation characteristic from relativistic motion**
3. Use the **Doppler-effect** to convert into the observer's frame of reference.
4. **Integrate over electron distribution** to obtain the final spectrum.

It is possible to do the same analytically without any approximations, however, we will use an approximate way here that is good enough to give the exact answer.

Synchrotron radiation (=Magnetobremstrahlung) is the energy radiated by charged particles moving in magnetic fields. We already know how to compute the energy loss via Larmor's formula, so what remains to do is to compute the acceleration that a particle has in a magnetic field.

We have to start, therefore, by looking at the motion of (relativistic) charges q in magnetic fields.

Assuming no electric field is present (which is the case, given that the universe is charge neutral, i.e., we can assume the same number of electrons and protons present), this motion is described by the Lorentz-Force

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v} \times \mathbf{B} \quad \text{where} \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\beta^2}} = \gamma m\mathbf{v} \quad \text{and} \quad \beta = v/c. \quad (6.1)$$

We will assume that there are no radiative losses, i.e., the only force the charge feels is due to the magnetic field. This is obviously not true on long timescales, because the particle is radiating synchrotron radiation, but it is generally true on short timescales, as we shall see later.

The velocity vector of the particle can be written in a component parallel to the magnetic field, \mathbf{v}_{\parallel} , and in a component \mathbf{v}_{\perp} perpendicular to the field:

$$\mathbf{v}_{\parallel} = \frac{\mathbf{v} \cdot \mathbf{B}}{B} \frac{\mathbf{B}}{B} \quad \mathbf{v}_{\perp} = \frac{\mathbf{B} \times (\mathbf{v} \times \mathbf{B})}{B^2} \quad (6.2)$$

$$|\mathbf{v}_{\parallel}| = v_{\parallel} = v \cos \alpha \quad |\mathbf{v}_{\perp}| = v_{\perp} = v \sin \alpha \quad (6.3)$$

where α , the pitch-angle, is $\angle(\mathbf{v}, \mathbf{B})$

Since

$$\mathbf{v} \times \mathbf{B} = (\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}) \times \mathbf{B} = \mathbf{v}_{\perp} \times \mathbf{B} \quad (6.4)$$

there is no acceleration parallel to the B -field and we can ignore v_{\parallel} for the moment. Since the force is perpendicular to \mathbf{v}_{\perp} and \mathbf{B} this means that we obtain the equations for circular motion around the magnetic field:

$$ma_{\perp} = \frac{\gamma m v_{\perp}^2}{R} = qv_{\perp}B \quad (6.5)$$

such that

$$\frac{v_{\perp}}{R} = \frac{1}{\gamma} \frac{qB}{m} \quad (6.6)$$

The period to go around a circle at velocity v is

$$P = \frac{2\pi R}{v} \quad (6.7)$$

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corresponding to a frequency

$$\omega = \frac{2\pi}{P} = \frac{v}{R} \quad (6.8)$$

Therefore we find the frequency with which the charged particle moves around the B -field is given by

$$\omega_B = \frac{1}{\gamma} \frac{qB}{m} = \frac{1}{\gamma} \omega_L \quad (6.9)$$

where

$$\omega_L = \frac{qB}{m} \iff \nu_L = \frac{\Omega_L}{2\pi} = \frac{qB}{2\pi m} \quad (6.10)$$

is called the *Larmor frequency* (also called the cyclo-frequency or the gyrofrequency).

The radius of the orbit of the particle, the *Larmor radius*, is

$$R = \frac{\gamma v_{\perp}}{\Omega_L} \quad (6.11)$$

To get a feel for some typical orders of magnitude, let's look at cosmic rays in the interstellar medium. These are produced, e.g., by acceleration of particles in supernova shock fronts. Instead of using the γ -factor of the particles as above, it is common to use their energy, E . Special relativity shows that the energy and γ are related by

$$E = \gamma mc^2 \quad (6.12)$$

Therefore, the Larmor radius is

$$R = \frac{\gamma v m}{qB} = \frac{\gamma mc^2}{qBc} = \frac{1}{qc} \frac{E}{B} \quad (6.13)$$

For cosmic rays, typical energies are $E = 1 \text{ GeV} = 1.6 \times 10^{-10} \text{ J}$ and a typical magnetic field in our milky way is $B = 10^{-10} \text{ T}$. With these values we obtain $R = 3 \times 10^{11} \text{ m} \sim 2 \text{ AU}$. This means that the radii are very small compared to typical length scales in the galaxy (which are measured in 10s of parsecs, where $1 \text{ pc} = 206525 \text{ AU}$). Therefore, we can say that cosmic rays are "frozen" into the magnetic field of our Galaxy and move "along" the magnetic field lines.

Radiated Energy

Electrodynamics: **Radiation of an accelerated electron:**

$$P_{\text{em}} = \frac{q^2}{6\pi c^3 \epsilon_0} \gamma^4 \left(\dot{v}_\perp^2 + \gamma^2 \dot{v}_\parallel^2 \right) \quad (6.14)$$

Derivation by Lorentz-Transforming the classical formula for dipole radiation ($P = 2e^2/3c^3 \cdot a^3$). Messy.

In case of **circular motion**, $\dot{v}_\perp = \omega_B v_\perp$. Hence

$$P_{\text{em}} = \frac{q^2}{6\pi c^3 \epsilon_0} \gamma^4 \frac{v_\perp^2 q^2 B^2}{\gamma^2 m^2} = 2\beta^2 \gamma^2 c \cdot \sigma_T \cdot U_B \cdot \sin^2 \alpha \quad (6.15)$$

where (for electrons, i.e., $q = e$)

$$U_B = B^2/2\mu_0 \quad (\text{Energy density of the } B\text{-field}), \quad (6.16)$$

$$\sigma_T = \frac{\mu_0 e^4}{6\pi m_e^2 \epsilon_0 c^2} \quad (\text{Thomson-cross section}) \quad (6.17)$$

Presence of σ_T due to quantum electrodynamics: Derivation of synchrotron-radiation in frame of reference of electron via interaction of electron with a virtual photon of the magnetic field (i.e., **Compton scattering with virtual photon**).

Radiated Energy

Radiated Energy

Total energy radiated: Integration over all electrons.

Assumption: Isotropic velocity distribution.

Average pitch angle

$$\langle \sin^2 \alpha \rangle = \frac{1}{4\pi} \int_0^{4\pi} \sin^2 \alpha d\Omega = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \sin^2 \alpha \sin \alpha d\alpha d\varphi = \frac{2}{3} \quad (6.18)$$

therefore

$$\langle P_{\text{em}} \rangle = \frac{4}{3} \beta^2 \gamma^2 c \sigma_T U_B \quad (6.19)$$

for $\beta \rightarrow 1$.

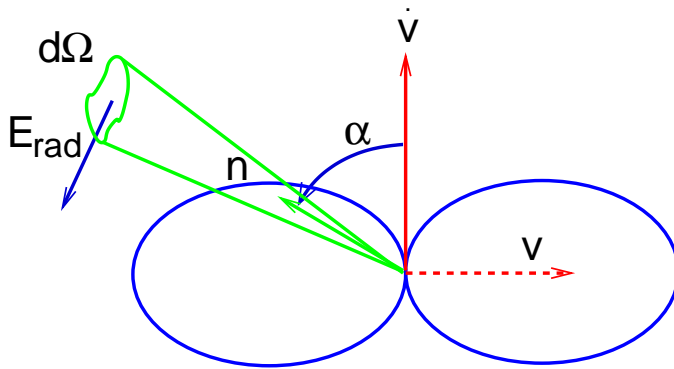
Note: Since $E = \gamma mc^2 \Rightarrow P \propto E^2 U_B$.

Note: $P_{\text{em}} \propto \sigma_T \propto m^{-2} \Rightarrow$ **Synchrotron radiation from charged particles with larger mass (protons, ...) is negligible.**

Note: Life-time of particle of energy E is

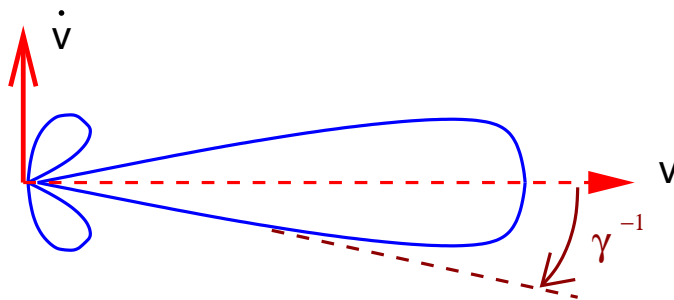
$$t_{1/2} \sim \frac{E}{P} \propto 1/(B^2 E) = 5 \text{ s} \left(\frac{B}{1 \text{ T}} \right)^{-2} \gamma^{-1} = 1.6 \times 10^7 \text{ years} \left(\frac{B}{10^{-7} \text{ T}} \right)^{-2} \gamma^{-1} \quad (6.20)$$

Single Electron spectrum, I



Frame of reference of electron:
Emitted radiation has **dipole characteristic** (see, e.g., Eq. 5.7).

after Rybicki & Lightman, Fig. 3.5

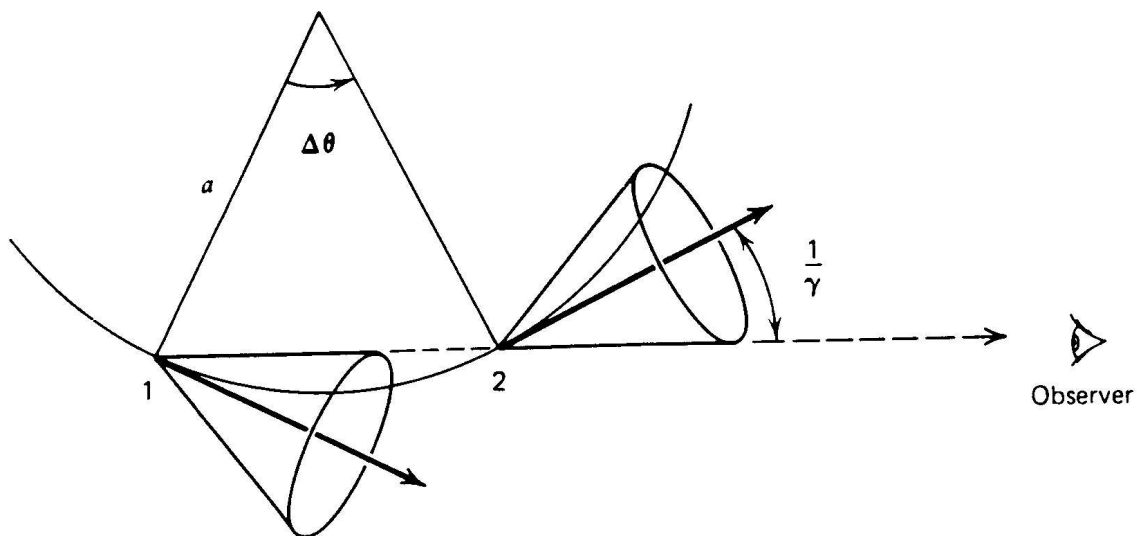


Lorentz-Transform into laboratory system: **Forward Beaming**. Opening angle is $\Delta\theta \approx \gamma^{-1}$.

after Rybicki & Lightman, Fig. 4.11d

Emitted spectrum

Single Electron spectrum, II



Rybicki & Lightman, Fig. 6.2

In the electron frame of rest, beam passes observer during time

$$\Delta t = \frac{\Delta\theta}{\omega_B} = \frac{m_e c \gamma}{eB} \frac{2}{\gamma} = \frac{2}{\omega_L} \quad (6.21)$$

Emitted spectrum

Single Electron spectrum, III

But: Doppler effect shortens duration of pulse (electron is closer to observer at end of beam).

⇒ Duration of pulse:

$$\tau = \left(1 - \frac{v}{c}\right) \Delta t = (1 - \beta) \Delta t \quad (6.22)$$

For $\gamma \gg 1$, i.e., $\beta = v/c \sim 1$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = (1 + \beta)(1 - \beta) \sim 2(1 - \beta) \quad (6.23)$$

such that

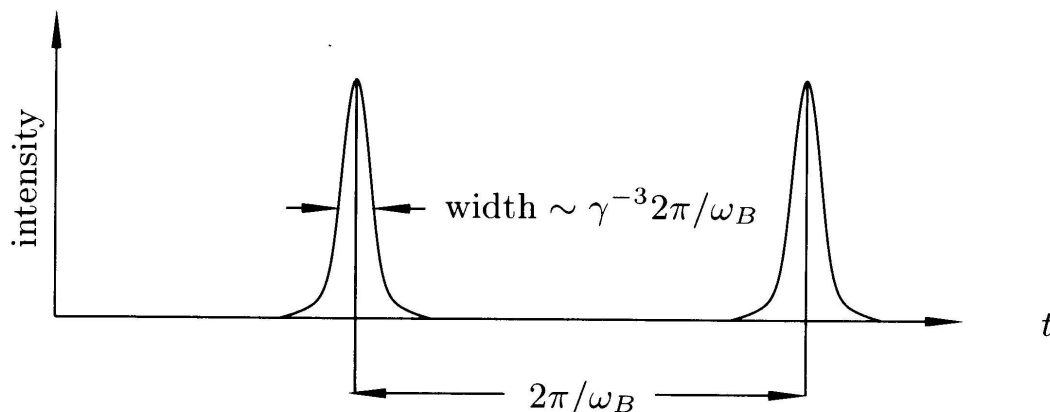
$$\tau = \frac{1}{2} \left(1 - \frac{v^2}{c^2}\right) \Delta t = \frac{1}{\gamma^2 \omega_L} \quad (6.24)$$

Thus the **characteristic frequency** of the radiation is given by

$$\omega_c = \gamma^2 \omega_L = \frac{eB}{m_e c} \left(\frac{E}{m_e c^2}\right)^2 \quad (6.25)$$

Emitted spectrum

Resulting Field



Shu, Fig. 18.2

The observed time-dependent E -Field, $E(t)$, from one electron is a sequence of pulses of width τ , separated by Δt .

Can approximate to good precision these single peaks by δ -functions.

In reality: Derive spectrum by **Fourier-transforming** $E(t)$. Basic result is the same.

Emitted spectrum

Power-law distribution, I

Spectral energy distribution P_ν of an electron with total energy $E = \gamma m_e c^2$ as

$$P_\nu(\gamma) = \frac{4}{3} \beta^2 \gamma^2 c \sigma_T U_B \delta(\nu - \gamma^2 \nu_L) \quad (6.26)$$

where $\delta(x)$ is a δ -function, i.e.,

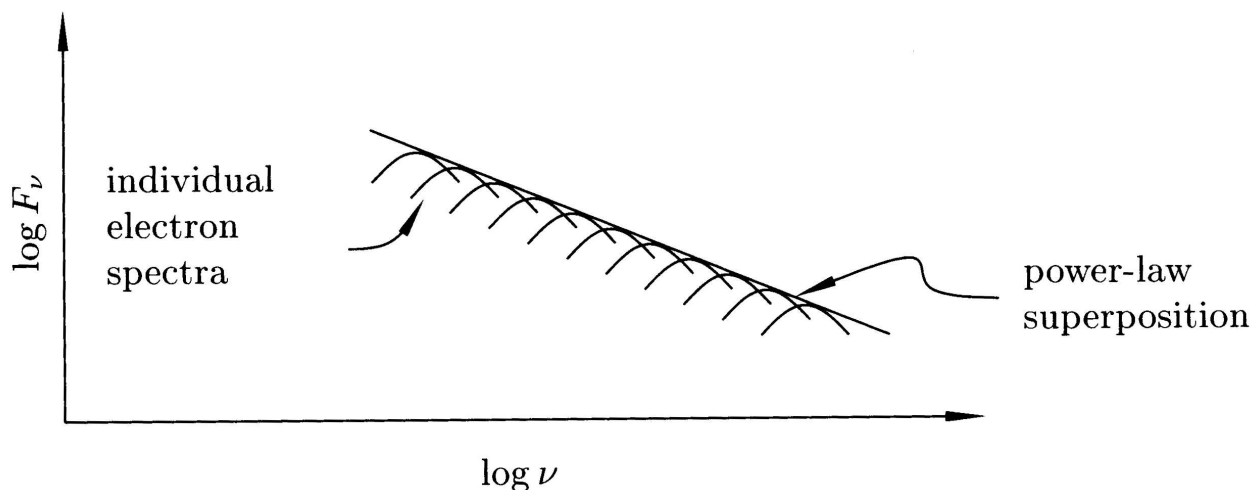
$$\delta(x) = 0 \quad \text{for } x \neq 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x) dx = 1 \quad (6.27)$$

i.e., electron with energy $\gamma m_e c^2$ "blinks" at frequency $\nu = \gamma^2 \nu_L$.

By definition, for **nonthermal synchrotron radiation**, electrons have a **power-law distribution**

$$n(\gamma) d\gamma = n_0 \gamma^{-p} d\gamma \quad (6.28)$$

Power-law distribution, II



Power emitted by electron distribution (=spectrum) found by integrating over all electrons

$$P_\nu = \int_1^\infty P_\nu(\gamma) n(\gamma) d\gamma \quad (6.29)$$

Power-law distribution, III

Therefore

$$P_\nu = \int_1^\infty \frac{4}{3} \beta^2 \gamma^2 c \sigma_T U_B \delta(\nu - \gamma^2 \nu_L) n_0 \gamma^{-p} d\gamma \quad (6.30)$$

since $\gamma \gg 1$: $\beta \approx 1$

$$= A \int_1^\infty \gamma^{2-p} \delta(\nu - \gamma^2 \nu_L) d\gamma \quad (6.31)$$

substituting $\nu' = \gamma^2 \nu_L$, i.e., $d\nu' = 2\nu_L \gamma d\gamma$

$$= B \int_{\nu_L}^\infty \gamma^{1-p} \delta(\nu - \nu') d\nu' \quad (6.32)$$

since $\gamma = (\nu/\nu_L)^{1/2}$, one finally finds

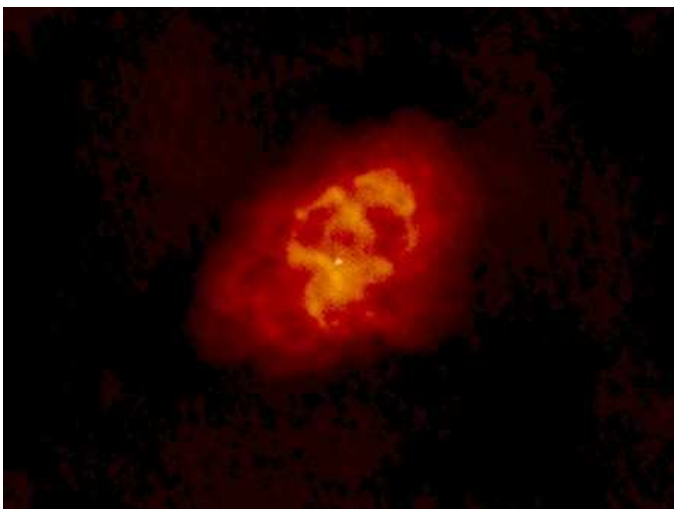
$$P_\nu = \frac{2}{3} c \sigma_T n_0 \frac{U_B}{\nu_L} \left(\frac{\nu}{\nu_L} \right)^{-\frac{p-1}{2}} \quad (6.33)$$

The spectrum of an electron power-law distribution is a power-law!

Application of Synchrotron Radiation



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Crab nebula at 90 cm (NRAO 300' telescope), resolution 1.3''

The characteristic frequency was

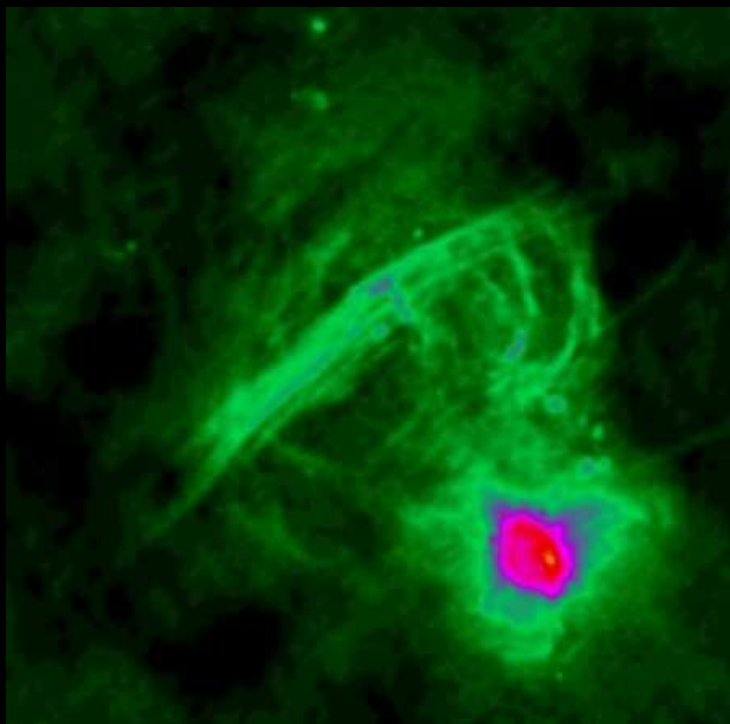
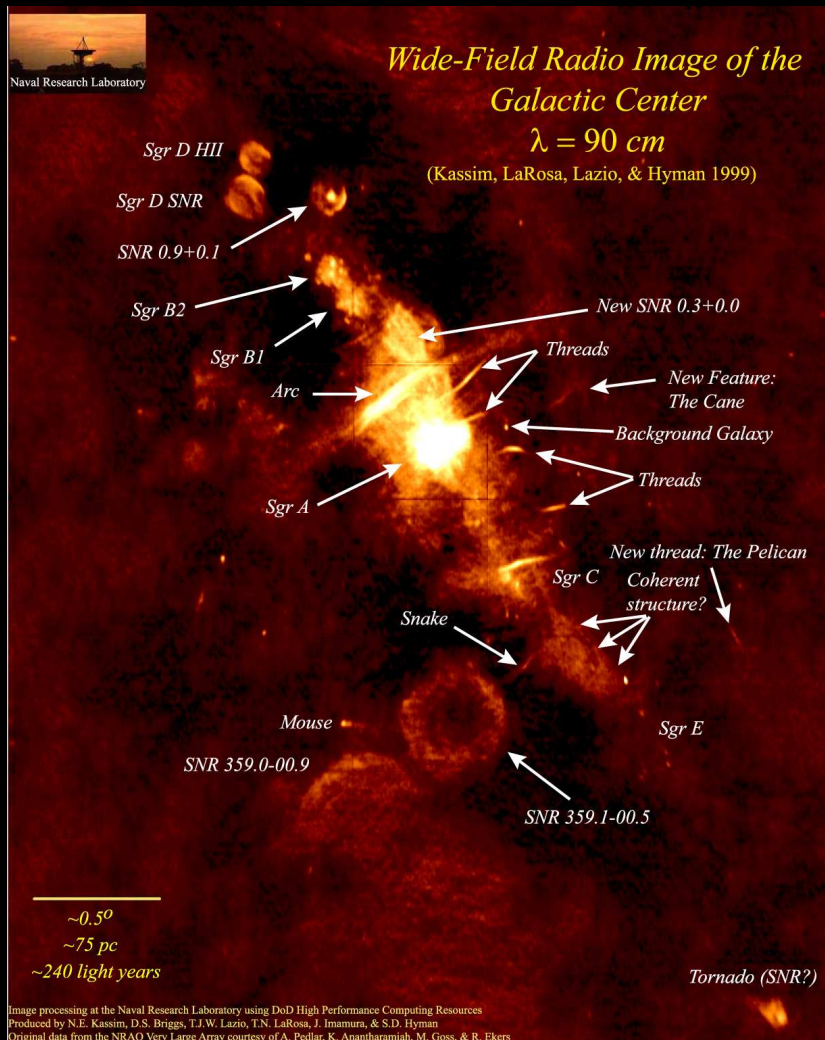
$$\nu_c = \frac{\omega_c}{2\pi} \quad (6.25)$$

$$= 6300 \left(\frac{B}{10^{-7} \text{ T}} \right) \left(\frac{E/m_e c^2}{10^3} \right)^2 \text{ MHz} \quad (7.1)$$

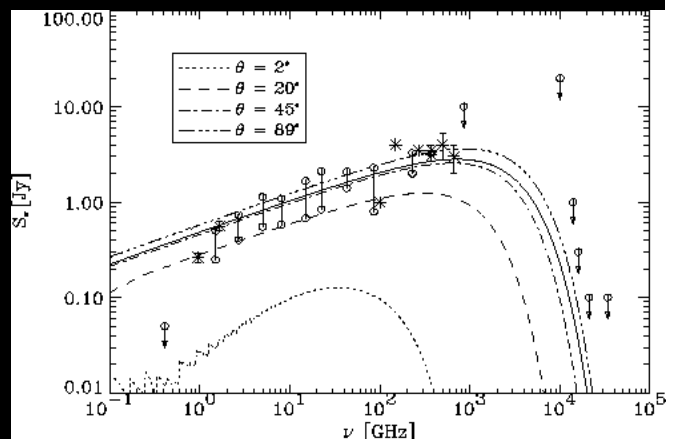
Optical light has $\nu \sim 10^8$ MHz. To emit this frequency, the electrons must have $\gamma \sim 10^6$ for a typical B -field!

Life time of electrons with $\gamma = 10^6$ (per Eq. 6.20): **16 years**.

Diameter of Crab: ~ 2 pc \implies it is not a problem to deliver all energy by accelerating electrons at the center of the neutron star in the center of the nebula.



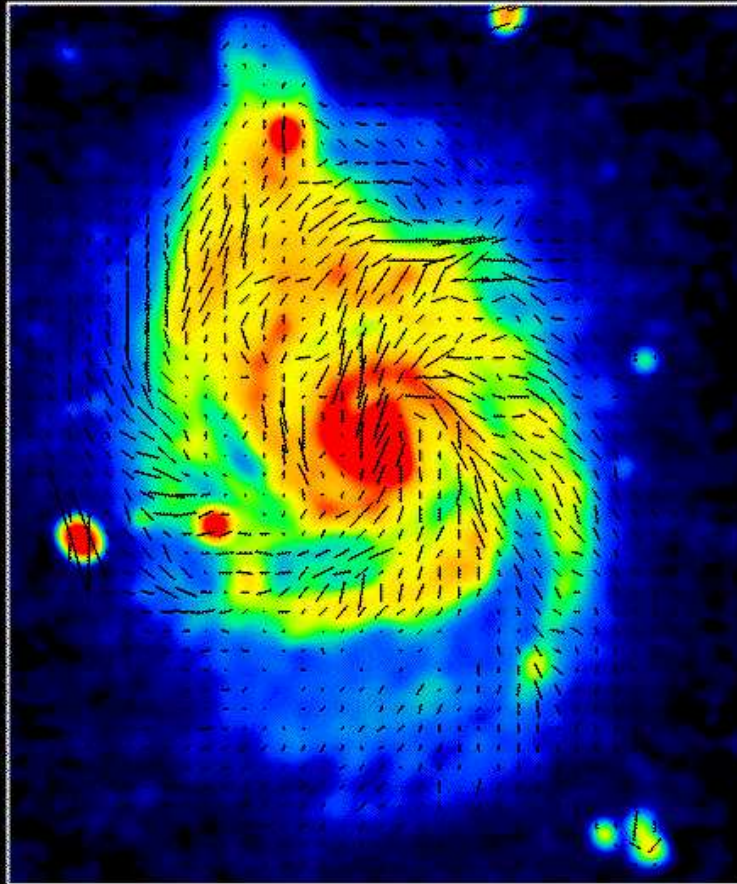
Galactic Center, courtesy NRAO



Emission from Sgr A, the galactic center: spectrum characteristic for synchrotron radiation.

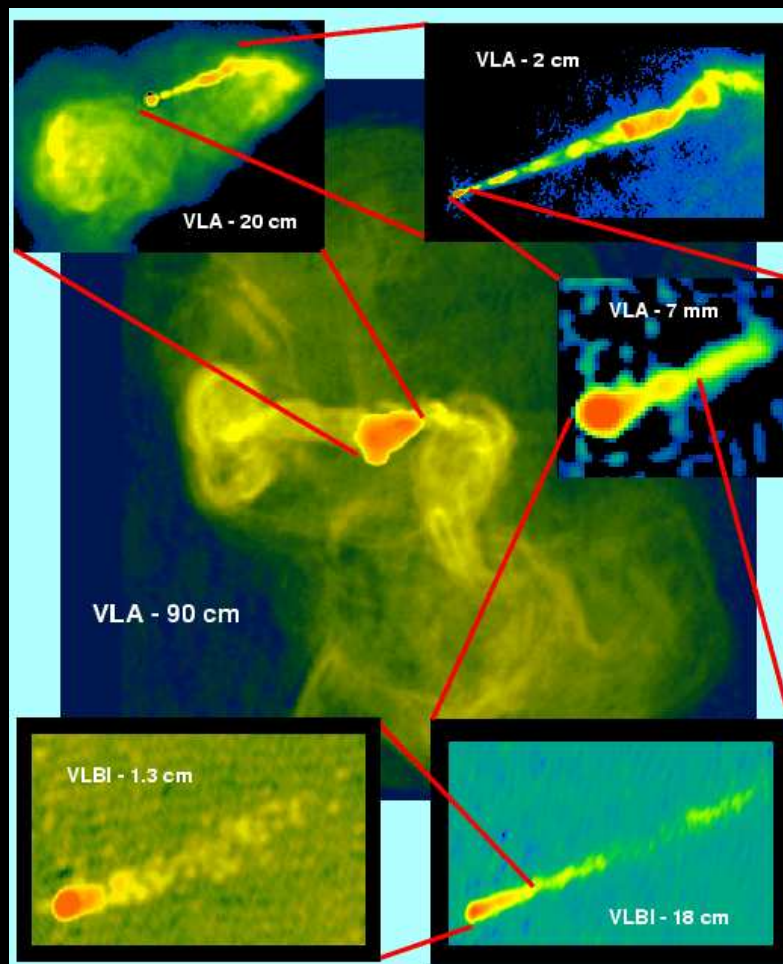
Note how emissivity follows B -field structure!

M51 20cm Total Intensity+Magnetic Field (VLA)



Copyright: MPIfR, Bonn (R.Beck, C.Horellou & N.Neinger)

B-field vectors inferred from the degree of polarization in spiral galaxy M51 by rotation of the observed *E*-field-vectors by 90° (Neinger 1992, A&A 263, 30)



AGN M87; courtesy Frazer Owen