Comptonization

Introduction

Comptonization: Upscattering of low-energy photons by inverse Compton collisions in a hot electron gas.

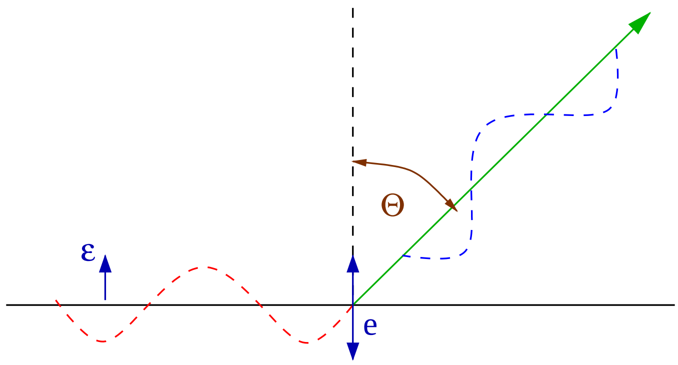
Astronomically important in

- galactic black hole candidates
- active galactic nuclei

Structure:

1. Scattering of photons off stationary electrons (Thomson scattering)
2. Quantum mechanical analogue (Compton scattering).
3. Scattering off nonstationary electrons
4. Results of detailed theory

Thomson Scattering



after Rybicki&Lightman, Fig. 3.6

Look at radiation from free electron in response to excitation of electron by an electromagnetic wave $E_0 \sin\omega_0 t$ (pointing in direction of unit-vector $\boldsymbol{\epsilon}$):
Force on charge

$$\mathbf{F} = m_e \dot{\mathbf{v}} = q E_0 \sin\omega_0 t\, \boldsymbol{\epsilon} \tag{8.1}$$

This neglects the B-field, i.e., assumes $v \ll c$.

$\Longrightarrow$ The electron feels an acceleration, $\dot{\mathbf{v}}$, and therefore it radiates!

Thomson Scattering

Larmor's formula gives the power radiated through the spherical angle $d\Omega$ in direction Θ:

$$\frac{dP}{d\Omega}(\Theta) = \frac{1}{16\pi^2 c^3 \epsilon_0} q^2 \dot{v}^2 \sin^2\Theta \quad \text{and (avg. over } \Omega) \quad P = \frac{q^2 \dot{v}^2}{6\pi c^3 \epsilon_0} \tag{8.2}$$

This follows from Eq. (5.7), which gives the flux through an area element $dA = r^2 d\Omega$.

Inserting $E(t)$ gives

$$\frac{dP}{d\Omega}(t) = \frac{q^2}{16\pi^2 c^3 \epsilon_0} \frac{q^2 E_0^2}{m^2} \sin^2\omega_0 t \sin^2\Theta \quad \text{and} \quad P(t) = \frac{q^2}{6\pi c^3 \epsilon_0} \frac{q^2 E_0^2}{m^2} \sin^2\omega_0 t \tag{8.3}$$

To obtain the average power emitted: average over time ($\langle \sin^2\omega_0 t \rangle = 1/2$)

$$\frac{dP}{d\Omega} = \frac{q^4 E_0^2}{16\pi^2 m^2 c^3 \epsilon_0} \sin^2\Theta \quad \text{and} \quad P = \frac{q^4 E_0^2}{12\pi c^3 m^2 \epsilon_0} \tag{8.4}$$

Note that the scattering angle is Θ, not θ. The reason for this will become clear shortly.

Thomson Scattering

The incident radiation flux on the electron (i.e., $c \times$ energy density for radiation)

$$\langle \mathbf{S} \rangle = \frac{c\epsilon_0}{2} E_0^2 \tag{8.5}$$

We define the differential cross section for Thomson scattering, $d\sigma/d\Omega$, such that

$$\frac{dP}{d\Omega} = \langle \mathbf{S} \rangle \frac{d\sigma}{d\Omega} \quad \Longleftrightarrow \quad \frac{q^4 E_0^2}{16\pi^2 m^2 c^3 \epsilon_0} \sin^2\Theta = \frac{c\epsilon_0^2}{2} E_0^2 \frac{d\sigma}{d\Omega} \tag{8.6}$$

such that

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{polarized}} = \frac{q^4}{8\pi^2 m^2 c^4 \epsilon_0^2} \sin^2\Theta = r_0^2 \sin^2\Theta \tag{8.7}$$

with the classical electron radius

$$r_0 = \frac{e^2}{4\pi m_\mathrm{e} c^2 \epsilon_0} = 2.82 \times 10^{-15}\,\mathrm{m} \tag{8.8}$$

Visualization: $d\sigma/d\Omega$ is the area presented by the electron to a photon that is going to get scattered in direction $d\Omega$.

Thomson Scattering

An identical derivation yields the total cross section for Thomson scattering, defined via

$$P = \langle S \rangle \,\sigma \tag{8.9}$$

to obtain

$$\sigma = \frac{8\pi}{3} r_0^2 =: \sigma_\mathrm{T} \tag{8.10}$$

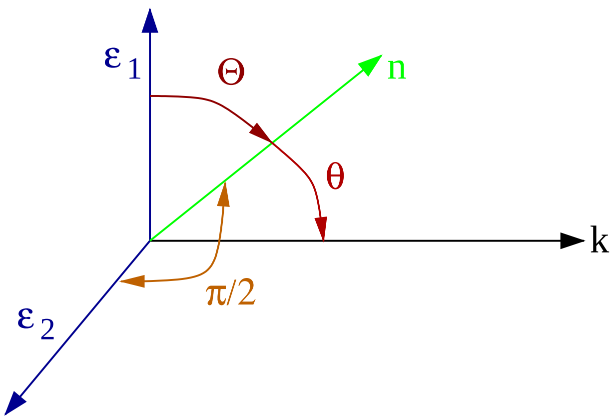
where

$$\sigma_\mathrm{T} = \frac{e^4}{6\pi m_\mathrm{e}^2 \epsilon_0^2 c^4} = 6.652 \times 10^{-29}\,\mathrm{m}^2 \tag{8.11}$$

(Thomson cross section)

Previous versions of σ_T used in these lectures are identical to the above if you strategically make use of $\epsilon_0 \mu_0 = c^2$!

Thomson Scattering



after Rybicki & Lightman, Fig. 3.7

For linear polarized light: scattered radiation is linearly polarized in direction of incident polarization vector, $\boldsymbol{\epsilon}$, and direction of scattering, $\mathbf{n}$.

To compute σ for nonpolarized radiation, note:

nonpolarized radiation $= \sum$ polarized beams at $\angle$

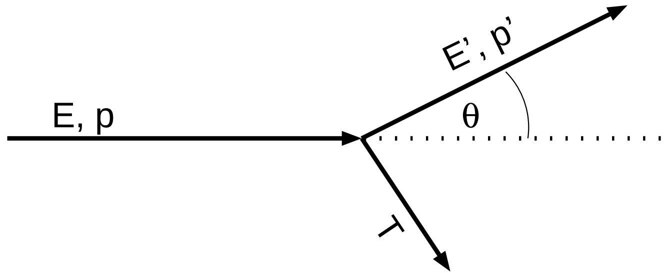
Thus, to scatter nonpolarized radiation propagating in direction $\mathbf{k}$ into direction $\mathbf{n}$, need to average two scatterings:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{unpol}} = \frac{1}{2}\left(\left.\frac{d\sigma(\Theta)}{d\Omega}\right|_{\text{pol}} + \left.\frac{d\sigma(\pi/2)}{d\Omega}\right|_{\text{pol}}\right) \quad (8.12)$$

Let $\theta = \angle(\mathbf{k}, \mathbf{n})$ to obtain

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{unpol}} = \frac{r_0^2}{2}(1+\cos^2\theta) = \frac{3\sigma_{\text{T}}}{16\pi}(1+\cos^2\theta) \quad \text{and} \quad \int \frac{d\sigma}{d\Omega}d\Omega = \sigma_{\text{T}} \quad (8.13)$$

Compton Scattering



Thomson scattering: initial and final wavelength identical.
But: in reality: light consists of photons
$\Longrightarrow$ Scattering: photon changes direction
$\Longrightarrow$ Momentum change
$\Longrightarrow$ Energy change!
This is quantum picture $\Longrightarrow$ Compton scattering.

Energy/wavelength change (see handout):

$$E' = \frac{E}{1 + \frac{E}{m_e c}(1-\cos\theta)} \sim E\left(1 - \frac{E}{m_e c}(1-\cos\theta)\right) \quad (8.14)$$

$$\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta) \quad (8.15)$$

where $h/m_e c = 2.426 \times 10^{-12}$ m (Compton wavelength).
Averaging over θ, for $E \ll m_e c$:

$$\frac{\Delta E}{E} \approx -\frac{E}{m_e c} \quad (8.16)$$

E.g., at 6.4 keV, $\Delta E \approx 0.2$ keV.

The following derivation will not be assessed (but you should know the end result!

The derivation of Eq. (8.14) is most simply done in special relativity using four-vectors. In the following, we will use capital letters for four-vectors and small letters for three-vectors. Furthermore, we will adopt the convention

$$\mathbf{P}\cdot\mathbf{Q} = P_0Q_0 - P_1Q_1 - P_2Q_2 - P_3Q_3 \tag{8.17}$$

for the product of two four vectors, following, e.g., the convention of Rindler (1991, Introduction to Special Relativity).

The four-momentum of a particle with non-zero rest-mass, m_0, e.g., an electron, is

$$\mathbf{Q} = m_0\gamma\begin{pmatrix} c \\ \mathbf{v}\end{pmatrix} = \begin{pmatrix} m_0\gamma c \\ \mathbf{q}\end{pmatrix} \tag{8.18}$$

where $\mathbf{v}$ is the velocity of the particle and $\mathbf{q}$ its momentum. As usual, $\gamma = (1-(v/c)^2)^{-1/2}$. The square of $\mathbf{Q}$ is

$$\mathbf{Q}^2 = m_0^2\gamma^2c^2 - m_0^2\gamma^2v^2 = m_0^2c^2\gamma^2\left(1-\left(\frac{v^2}{c^2}\right)\right) = m_0^2c^2 \tag{8.19}$$

Obviously, $\mathbf{Q}^2$ is relativistically invariant.

In the same spirit, the four-momentum of a photon is

$$\mathbf{P} = \frac{E}{c}\begin{pmatrix} 1 \\ \hat{\mathbf{u}}\end{pmatrix} \tag{8.20}$$

where $\hat{\mathbf{u}}$ is an unit-vector pointing into the direction of motion of the photon. Note that for photons

$$\mathbf{P}^2 = 0 \tag{8.21}$$

as the photon's rest-mass is zero.

We will now look at the collision between a photon and an electron. We will denote the four-momenta after the collision with primed quantities.

Conservation of four-momentum requires

$$\mathbf{P}+\mathbf{Q} = \mathbf{P}'+\mathbf{Q}' \tag{8.22}$$

We now use a trick from Lightman et al. (1975, Problem Book in Relativity and Gravitation), solving this equation for $\mathbf{Q}'$ and squaring the resulting expression:

$$(\mathbf{P}+\mathbf{Q}-\mathbf{P}')^2 = (\mathbf{Q}')^2 \tag{8.23}$$

Since the collision is elastic, i.e., the rest mass of the electron is not changed by the collision,

$$\mathbf{Q}^2 = (\mathbf{Q}')^2 \tag{8.24}$$

furthermore, $\mathbf{P}^2 = (\mathbf{P}')^2 = 0$, such that

$$\mathbf{P}\cdot\mathbf{Q} - \mathbf{P}\cdot\mathbf{P}' - \mathbf{Q}\cdot\mathbf{P}' = 0 \qquad\Longleftrightarrow\qquad \mathbf{P}\cdot\mathbf{P}' = \mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}') \tag{8.25}$$

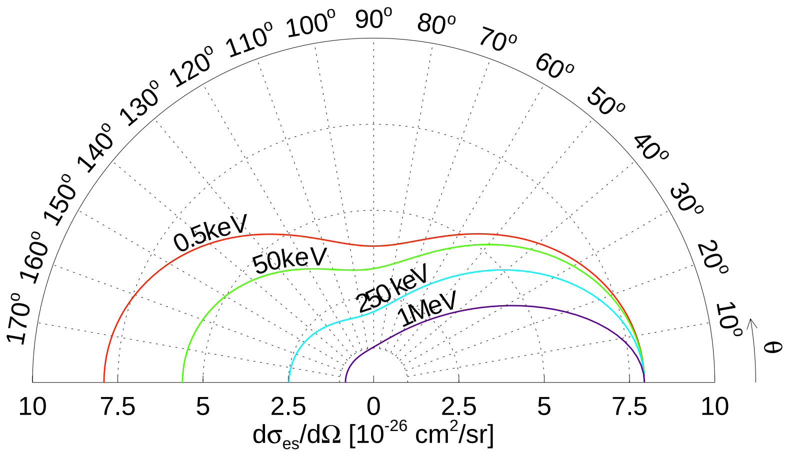
But in the frame where the electron is initially at rest,

$$\mathbf{Q}\cdot(\mathbf{P}-\mathbf{P}') = m_\mathrm{e}c\left(\frac{E}{c}-\frac{E'}{c}\right) = m(E-E') \tag{8.26}$$

$$\mathbf{P}\cdot\mathbf{P}' = \frac{E}{c}\frac{E'}{c}(1-\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}') = \frac{EE'}{c^2}(1-\cos\theta) \tag{8.27}$$

where $\theta = \angle(\hat{\mathbf{u}},\hat{\mathbf{u}}')$. Inserting into Eq. (8.25) and solving for E' gives Eq. (8.14).

Compton Scattering



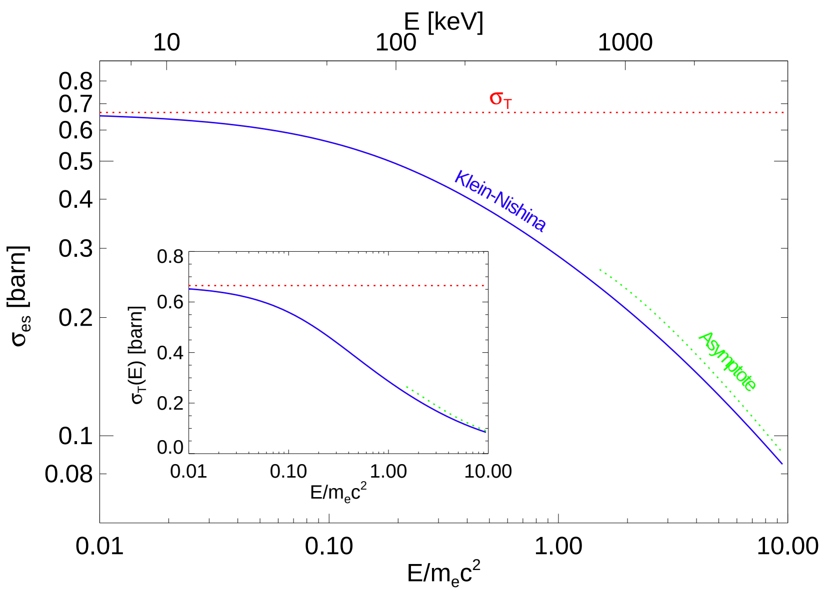
Proper derivation of cross section done in quantum electrodynamics.
In the limit of low energies: will find Thomson result, for higher energies: relativistic effects become important.

For unpolarized radiation,

$$\frac{d\sigma_{\mathrm{es}}}{d\Omega} = \frac{3}{16\pi}\sigma_{\mathrm{T}}\left(\frac{E'}{E}\right)^2\left(\frac{E}{E'}+\frac{E'}{E}-\sin^2\theta\right) \tag{8.28}$$

(Klein-Nishina formula).

Compton Scattering



1 barn $= 10^{-28}$ m^2

Integrating over $d\sigma_{\mathrm{es}}/d\Omega$ gives total cross-section:

$$\sigma_{\mathrm{es}} = \frac{3}{4}\sigma_{\mathrm{T}}\left[\frac{1+x}{x^3}\left\{\frac{2x(1+x)}{1+2x}-\ln(1+2x)\right\}+\frac{1}{2x}\ln(1+2x)-\frac{1+3x}{(1+2x)^2}\right] \tag{8.29}$$

where $x = E/m_{\mathrm{e}}c^2$.

Energy Exchange

For non-stationary electrons, use previous formulae and Lorentz transform photon into electron's frame of rest (FoR):

1. Lab system $\Rightarrow$ electron's frame of rest:
$$E_{\mathrm{FoR}} = E_{\mathrm{Lab}}\gamma(1 - \beta\cos\theta) \quad (8.30)$$
2. Scattering occurs, gives E'_{FoR}.
3. Electron's frame of rest $\Rightarrow$ Lab system:
$$E'_{\mathrm{Lab}} = E'_{\mathrm{FoR}}\gamma(1 + \beta\cos\theta') \quad (8.31)$$

Therefore, if electron is relativistic:

$$E'_{\mathrm{Lab}} \sim \gamma^2 E_{\mathrm{Lab}} \quad (8.32)$$

since (on average) θ, θ' are $\mathcal{O}(\pi/2)$ (beaming!).
Thus: Energy transfer is *very* efficient.

Compton catastrophe

One can show (see handout) that the net power gained by photons scattering off monoenergetic electrons with gamma-factor γ is

$$P_{\mathrm{compt}} = \frac{4}{3}\sigma_{\mathrm{T}} c\gamma^2\beta^2 U_{\mathrm{rad}} \quad (8.33)$$

where U_{rad} is the energy density of the photon field (see handout).
But the power emitted by synchrotron radiation in a B-field of energy density U_{B} was

$$P_{\mathrm{synch}} = \frac{4}{3}\sigma_{\mathrm{T}} c\gamma^2\beta^2 U_{\mathrm{B}} \quad (6.19)$$

Magnetized plasma: synchrotron photons are inverse Compton scattered by the electrons. Ratio of emitted powers:

$$\frac{P_{\mathrm{compt}}}{P_{\mathrm{synch}}} = \frac{U_{\mathrm{rad}}}{U_{\mathrm{B}}} \quad (8.34)$$

Consequence of the fact that (in QED) synchrotron radiation is inverse Compton scattering off virtual photons of the B-field.

For $U_{\mathrm{rad}} > U_{\mathrm{B}}$ this means $P_{\mathrm{compt}} > P_{\mathrm{synch}} \Longrightarrow$ (synchrotron) photon field will undergo dramatic amplification $\Longrightarrow$ very efficient cooling of electrons by inverse Compton losses (Compton catastrophe). $\Longrightarrow$ This defines a maximum brightness for any synchrotron emitting source.

The following derivation will not be assessed (but you should know its result!).

To derive Eq. (8.33), we first look at the energy budget of one single scattering.

The total power *emitted* in the frame of rest of the electron is given by

$$\left.\frac{dE'_{\mathrm{FoR}}}{dt_{\mathrm{FoR}}}\right|_{\mathrm{em}} = \int c\sigma_{\mathrm{T}} E'_{\mathrm{FoR}} V'(E'_{\mathrm{FoR}}) dE'_{\mathrm{FoR}} \tag{8.35}$$

where $V'(E')$ is the photon energy density distribution (number of photons per cubic metre with an energy between E' and $E' + dE'$).

One can show that is Lorentz invariant:

$$\frac{V_{\mathrm{Lab}}(E_{\mathrm{Lab}})dE_{\mathrm{Lab}}}{E_{\mathrm{Lab}}} = \frac{V_{\mathrm{FoR}}(E_{\mathrm{FoR}})dE_{\mathrm{FoR}}}{E_{\mathrm{FoR}}} \tag{8.36}$$

In the “Thomson limit” one assumes that the energy change of the photon in the rest frame of the electron is small,

$$E'_{\mathrm{FoR}} = E_{\mathrm{FoR}} \tag{8.37}$$

Furthermore one can show that the power is Lorentz invariant:

$$\frac{dE_{\mathrm{FoR}}}{dt_{\mathrm{FoR}}} = \frac{dE_{\mathrm{Lab}}}{dt_{\mathrm{Lab}}} \tag{8.38}$$

(this follows from the fact that the formulae for the Lorentz transform of Energy and time look the same).

Therefore

$$\left.\frac{dE_{\mathrm{Lab}}}{dt_{\mathrm{Lab}}}\right|_{\mathrm{em}} = c\sigma_{\mathrm{T}} \int E^2_{\mathrm{FoR}} \frac{V_{\mathrm{FoR}} dE_{\mathrm{FoR}}}{E_{\mathrm{FoR}}} \tag{8.39}$$

$$= c\sigma_{\mathrm{T}} \int E^2_{\mathrm{FoR}} \frac{V_{\mathrm{Lab}} dE_{\mathrm{Lab}}}{E_{\mathrm{Lab}}} \tag{8.40}$$

. . . Lorentz transforming E_{FoR}

$$= c\sigma_{\mathrm{T}}\gamma^2 \int (1 - \beta\cos\theta)^2 E_{\mathrm{Lab}} V_{\mathrm{Lab}} dE_{\mathrm{Lab}} \tag{8.41}$$

. . . averaging over angles ($\langle\cos\theta\rangle = 0$, $\langle\cos^2\theta\rangle = \frac{1}{3}$)

$$= c\sigma_{\mathrm{T}}\gamma^2 \left(1 + \frac{\beta^2}{3}\right) U_{\mathrm{rad}} \tag{8.42}$$

where

$$U_{\mathrm{rad}} = \int E V(E) dE \tag{8.43}$$

(initial photon energy density).

To determine the power gain of the photons, we need to subtract the power irradiated onto the electron,

$$\left.\frac{dE_{\mathrm{Lab}}}{dt_{\mathrm{Lab}}}\right|_{\mathrm{inc}} = c\sigma_{\mathrm{T}} \int E V(E) dE = \sigma_{\mathrm{T}} c U_{\mathrm{rad}} \tag{8.44}$$

Therefore, since

$$\gamma^2 - 1 = \gamma^2\beta^2 \tag{8.45}$$

the net power gain of the photon field is

$$P_{\mathrm{compt}} = \left.\frac{dE_{\mathrm{Lab}}}{dt}\right|_{\mathrm{em}} - \left.\frac{dE_{\mathrm{Lab}}}{dt}\right|_{\mathrm{inc}} \tag{8.46}$$

$$= \frac{4}{3}\sigma_{\mathrm{T}} c\gamma^2\beta^2 U_{\mathrm{rad}} \tag{8.47}$$

Amplification factor

In electron frame of rest,

$$\frac{\Delta E}{E} = -\frac{E}{m_\mathrm{e}c} \qquad (8.16)$$

Assuming a thermal (Maxwell) distribution of electrons (i.e., they're not at rest), one can show that the relative energy change is given by

$$\frac{\Delta E}{E} = \frac{4kT - E}{m_\mathrm{e}c} \qquad (8.48)$$

where A is the Compton amplification factor.
Thus:

$E \lesssim 4kT_\mathrm{e} \Longrightarrow$ Photons gain energy, gas cools down.
$E \gtrsim 4kT_\mathrm{e} \Longrightarrow$ Photons loose energy, gas heats up.

Amplification factor

In reality, photons will scatter more than once before leaving the hot electron medium.

The *total* relative energy change of photons by traversal of a hot ($E \ll kT_\mathrm{e}$) medium with electron density n_e and size ℓ is then approximately

$$\left(\text{rel. energy change } y\right) = \frac{\text{rel. energy change}}{\text{scattering}} \times \left(\text{\# scatterings}\right) \qquad (8.49)$$

The number of scatterings is $\max(\tau_\mathrm{e}, \tau_\mathrm{e}^2)$, where $\tau_\mathrm{e} = n_\mathrm{e}\sigma_\mathrm{T}\ell$ ("optical depth"), such that

$$y = \frac{4kT_\mathrm{e}}{m_\mathrm{e}c} \max(\tau_\mathrm{e}, \tau_\mathrm{e}^2) \qquad (8.50)$$

"Compton y-Parameter"

Amplification factor

The detailed derivation of the spectrum for Comptonization is difficult and is obtained by solving the non-relativistic diffusion equation for the motion of photons through phase-space (Kompaneets, 1957):

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x} x^4 \left(n + n^2 + \frac{\partial n}{\partial x} \right) \tag{8.51}$$

(Kompaneets equation) where

$$n = I(E)\frac{(hc)^2}{8\pi E^3} : \text{Photon Occupation Number}$$

$$x = E/kT_\mathrm{e} : \text{Photon energy}$$

$$y = \frac{4kT_\mathrm{e}}{m_\mathrm{e}c}\sigma_\mathrm{T} N_\mathrm{e} ct : \text{Kompaneets parameter}$$

Interpretation:

$$\partial n/\partial x : \text{Doppler-Motion}$$

$$n : \text{Recoil-Effect}$$

$$n^2 : \text{Induced/Stimulated emission}$$