Department of Physics
3rd Year Modules 2004/2005
Term 1, Week 01–05

THE UNIVERSITY OF WARWICK

PX318: Astrophysics from Space – Example Solutions

Academic Week 01: Accretion

Question 1: *Accretion in X-ray Binaries and Active Galactic Nuclei*
The derivation of the accretion luminosity shown in the lectures was somewhat simplified, as the kinetic energy of the accreted material was ignored. In this question you will check whether ignoring the kinetic energy was justified.

a) Using energy conservation arguments, derive a formula for the energy released by a mass m that is accreted from infinity to a distance r from a compact object of mass M. You may assume that m is stationary at infinity. At distance r, the mass is moving around the compact object in a circular orbit. (*N.B.:* For the purposes of this question it is sufficient to assume the Newtonian physics holds throughout, although properly spoken the general theory of relativity would have to be used; *Answer:* $E = GMm/2r$)

Solution: Because of energy conservation, the energy released by m when falling down to a distance r from the compact object is the difference between its gravitational potential energy at infinity (which is zero in the usual normalisation of the potential energy) and its total energy at distance r. The latter is the sum of the gravitational potential energy of m at distance r and of its kinetic energy:

$$E(r) = -\frac{GMm}{r} + \frac{1}{2}mv^2 \tag{s1.1}$$

The velocity on a circular orbit can be derived from balancing the centripetal force and the gravitational force:

$$G\frac{Mm}{r^2} = \frac{mv^2}{r} \quad \text{such that} \quad v = \sqrt{\frac{GM}{r}} \tag{s1.2}$$

Inserting v into Eq. (s1.1) gives

$$E(r) = -\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r} \tag{s1.3}$$

such that the energy released when material is falling from infinity to r is given by

$$E_{\text{released}}(r) = E(\infty) - E(r) = 0 - \left(-\frac{GMm}{2r}\right) = \frac{GMm}{2r} \tag{s1.4}$$

b) Use your result from the previous question to determine the energy released by one kilogram of material that is accreted from infinity onto the surface of a $1.44\,M_\odot$ neutron star with radius $r_{\text{NS}} = 10\,\text{km}$. To appreciate the magnitude of the energy released, give it not only in joules but also in Megatons TNT, where $1\,\text{MT} = 3 \times 10^{15}\,\text{J}$, and is a typical strength for todays nuclear bombs ($G = 6.67 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2}$, $1\,M_\odot = 2 \times 10^{30}\,\text{kg}$).

Solution: Inserting numbers into Eq. (s1.4) gives

$$E_{\text{released}} = \frac{6.67 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2} \cdot 1.44 \cdot 2 \times 10^{30}\,\text{kg} \cdot 1\,\text{kg}}{2 \cdot 10 \cdot 10^3\,\text{m}} = 9.6 \times 10^{15}\,\text{N}\,\text{m} = 9.6 \times 10^{15}\,\text{J} \sim 3\,\text{MT}$$

In other words: throwing 1 kg material onto a neutron star results in an energy release comparable to the explosion of a nuclear weapon.

c) As shown in the lectures the luminosity is defined by the power released by an astronomical object, i.e., by the energy released per time interval.

1. Convince yourself that this definition is equivalent to $L = \mathrm{d}E/\mathrm{d}t$ and use this equation to derive a formula for the total luminosity of an accreting black hole with a *mass accretion rate* $\dot{m}$, where $\dot{m} = \mathrm{d}m/\mathrm{d}t$ is the amount of mass accreted, $\mathrm{d}m$ per time interval $\mathrm{d}t$ (the total luminosity is the energy released by accretion to the Schwarzschild radius, $r_\mathrm{S} = 2GM/c^2$). Does the luminosity depend on the mass of the black hole?

Solution: Per definition of the luminosity,

$$L = \frac{\Delta E}{\Delta t} \tag{s1.5}$$

where ΔE is the energy released during the time interval Δt. Taking the limit $\Delta t \to 0$ then gives the desired definition.
Differentiating Eq. (s1.4) with respect to time gives

$$L = \frac{GM\dot{m}}{2r} \tag{s1.6}$$

Inserting the Schwarzschild radius for r then gives the luminosity for an accreting black hole

$$L_\mathrm{BH} = \frac{GM\dot{m}}{2 \cdot 2GM/c^2} = \frac{1}{4}\dot{m}c^2 \tag{s1.7}$$

which only depends on the mass accretion rate and not on the mass of the black hole (*N.B.* the correct relativistic equations show that $L_\mathrm{BH} \sim 0.1\dot{m}c^2$ for normal (“Schwarzschild”) black holes, and $L_\mathrm{BH} \sim 0.42\dot{m}c^2$ for maximally rotating “Kerr” black holes).

2. Estimate the mass accretion rate for a supermassive black hole in an Active Galactic Nucleus with a luminosity of $10^{13}\,L_\odot$. Appropriate units for your answer are solar masses per year ($L_\odot = 4 \times 10^{26}\,\mathrm{W}$, $c = 300000\,\mathrm{km\,s^{-1}}$, 1 year=365.25 days, 1 day=86400 seconds; note that your answer will differ from the numbers given in the lectures due to the different assumptions used here).

Solution: According to Eq. (s1.7),

$$\dot{m} = \frac{4L_\mathrm{BH}}{c^2} = \frac{4 \cdot 10^{13} \cdot 4 \times 10^{26}\,\mathrm{W}}{9 \times 10^{16}\,\mathrm{m^2\,s^{-2}}} = 1.78 \times 10^{23}\,\mathrm{kg\,s^{-1}} = 5.6 \times 10^{30}\,\mathrm{kg\,year^{-1}} = 2.8\,M_\odot\,\mathrm{year^{-1}} \tag{s1.8}$$

since $1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}} = 1\,\mathrm{m^2\,kg\,s^{-3}}$

Question 2: *Black body radiation*

We will only have very short time talking about black body radiation in the lectures, however, most of you will already have seen the definition of black body radiation and Planck’s radiation formula, e.g., in your quantum physics modules or in the previous astronomy lectures. If you have not, then now is a good time to read up on its major properties, some of which you will now be deriving.
Planck showed that the spectral energy distribution of a body in thermodynamic equilibrium (a “black body”) is given by

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1} \tag{2.1}$$

a) Convince yourself that the units of B_ν are $\mathrm{J\,m^{-2}\,s^{-1}\,Hz^{-1}\,sr^{-1}}$

Solution: The units of Planck’s constant h are J s, and therefore the units of B_ν are:

$$[B_\nu] = \frac{\mathrm{J\,s\,Hz^3}}{\mathrm{m^2\,s^{-2}}} = \frac{\mathrm{J}}{\mathrm{m^2}} = \frac{\mathrm{J}}{\mathrm{m^2\,s\,Hz\,sr^{-1}}} \tag{s2.1}$$

b) Convince yourself that the Planck function expressed in wavelengths, λ, is given by

$$B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT) - 1} \tag{2.2}$$

Note that you have to ensure energy conservation, i.e., $\int_0^\infty B_\lambda d\lambda = \int_0^\infty B_\nu d\nu$.

Solution: Energy conservation therefore requires

$$B_\lambda = B_\nu \left|\frac{d\nu}{d\lambda}\right| \tag{s2.2}$$

Since $\lambda\nu = c$,

$$\frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2} \tag{s2.3}$$

The remainder is simple algebra.

c) Show that for $h\nu \ll kT$ Planck's equation can be approximated by the *Rayleigh-Jeans Law*

$$B_\nu \approx \frac{2\nu^2}{c^2} kT \tag{2.4}$$

Solution: For $h\nu \ll kT$,

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + \ldots \tag{s2.4}$$

Inserting into Planck's law then gives the desired answer.

d) Show that the frequency of maximum intensity of B_ν is given by

$$h\nu_{\max} = 2.82 \cdot kT \tag{2.5}$$

This is Wien's displacement law. You may use the fact that the solution of the transcendent equation

$$x = 3(1 - \exp(-x)) \tag{2.6}$$

is $x \sim 2.82$ and can only be found numerically.

Solution: Wien's law follows from solving

$$\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu=\nu_{\max}} = 0 \tag{s2.5}$$

where

$$\frac{\partial B_\nu}{\partial \nu} = \frac{6h\nu^2}{c^2}\frac{1}{\exp(h\nu/kT) - 1} - \frac{2h2\nu^3}{kTc^2}\frac{\exp h\nu/kT}{(\exp(h\nu/kT) - 1)^2} \tag{s2.6}$$

which, after some algebra and after substituting $x = kT/h\nu$, yields Eq. (2.6).

Note that because of the different numerical form of B_λ, a similar excercise for $\lambda_{\max}$ results in a value for the wavelength of maximum flux such that $\lambda_{\max}\nu_{\max} \neq c$! This is caused by the energy conservation arguments used in subquestion a.

e) By integrating B_ν from 0 to ∞, show that the total power emitted from a black body per unit surface area is given by the Stefan-Boltzmann law,

$$P = \frac{\sigma}{\pi} T^4 \tag{2.7}$$

where the Stefan-Boltzmann constant is

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3} \tag{2.8}$$

You might want to make use of

$$\int_0^\infty \frac{x^3 dx}{\exp(x) - 1} = \frac{\pi^4}{15} \tag{2.9}$$

(this follows from a rather tricky application of the residue theorem for the integration of complex variables).

Solution: This question boils down to a straightforward integration:

$$P = \int_0^\infty B_\nu(T) d\nu \tag{s2.7}$$

. . . substituting $x = h\nu/kT$

$$= \frac{2h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{\exp(x) - 1} \tag{s2.8}$$

. . . as mentioned above, the integral has the value $\pi^4/15$

$$= \frac{2\pi^4 k^4}{15 c^2 h^3} T^4 = \frac{\sigma T^4}{\pi} \tag{s2.9}$$

Question 3: *Comments on this week's lectures*

In order to improve the teaching and to enable myself to react to problems you might have with the module, I would like to hear your opinion on my teaching as early as possible. I would appreciate it if you would voice any problems and criticisms as soon as possible, e.g., on the speed with which I talk about the subjects of the lectures, the overall difficulty level of the class and the homework, the quality and contents of the handouts, and so on.

Please write these comments on a separate sheet of paper and give them to me: Either put the paper on the lectern before class or put it in my "pigeon hole" in the mailboxes on the 5th floor of the physics building, close to the physics undergraduate office. Feel free to remain anonymous, if you deem this necessary. You can also ask questions by sending email to `j.wilms@warwick.ac.uk`.

Solutions to all questions can be found at `http://pulsar.astro.warwick.ac.uk/wilms/teach/astrospace/handouts.html`.