Department of Physics
3rd Year Modules 2004/2005
Term 1, Week 01–05

THE UNIVERSITY OF WARWICK

# PX318: Astrophysics from Space – Example Solutions

## Academic Week 03: Synchrotron Radiation

**Question 1:** *Synchrotron radiation: Derivation*
Using the outline shown in the lectures, show explicitly that the synchrotron spectrum of a power-law distribution of electrons is a power law and verify that the equation for the total synchrotron power emitted at frequency $\nu$ given in the lectures,

$$P_\nu = \frac{2}{3} c\sigma_\mathrm{T} n_0 \frac{U_\mathrm{B}}{\nu_\mathrm{L}} \left(\frac{\nu}{\nu_\mathrm{L}}\right)^{-\frac{p-1}{2}} \tag{1.1}$$

is correct.

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*Solution:* As shown in the lectures, the spectral energy distribution $P_\nu$ of an electron with total energy $E = \gamma m_\mathrm{e} c^2$ is given by

$$P_\nu(\gamma) = \frac{4}{3}\beta^2\gamma^2 c\sigma_\mathrm{T} U_\mathrm{B}\delta(\nu - \gamma^2\nu_\mathrm{L}) \tag{s1.1}$$

where $\delta(x)$ is the $\delta$-function, i.e.,

$$\delta(x) = 0 \quad \text{for } x \neq 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \tag{s1.2}$$

For a power law distribution, the number density of electrons at with $\gamma$-factor $\gamma$ is

$$n(\gamma)d\gamma = n_0\gamma^{-p}d\gamma \tag{s1.3}$$

The total emitted synchrotron spectrum of this electron population is obtained from summing over all energies

$$P_\nu = \int_1^\infty P_\nu(\gamma)n(\gamma)d\gamma \tag{s1.4}$$

Therefore

$$P_\nu = \int_1^\infty \frac{4}{3}\beta^2\gamma^2 c\sigma_\mathrm{T} U_\mathrm{B}\delta(\nu - \gamma^2\nu_\mathrm{L}) n_0\gamma^{-p}d\gamma \tag{s1.5}$$

since $\gamma \gg 1$ we can assume that $\beta \approx 1$ and therefore

$$= \frac{4}{3} c\sigma_\mathrm{T} U_\mathrm{B} n_0 \int_1^\infty \gamma^{2-p}\delta(\nu - \gamma^2\nu_\mathrm{L})d\gamma \tag{s1.6}$$

substituting $\nu' = \gamma^2\nu_\mathrm{L}$, i.e., $d\nu' = 2\nu_\mathrm{L}\gamma d\gamma$

$$= \frac{4}{3} c\sigma_\mathrm{T} U_\mathrm{B} n_0 \int_{\nu_\mathrm{L}}^\infty \frac{\gamma^{2-p}}{2\gamma\nu_\mathrm{L}}\delta(\nu - \nu')d\nu' \tag{s1.7}$$

which simplifies to

$$= \frac{2}{3} c\sigma_\mathrm{T} \frac{U_\mathrm{B}}{\nu_\mathrm{L}} n_0 \int_{\nu_\mathrm{L}}^\infty \gamma^{1-p}\delta(\nu - \nu')d\nu' \tag{s1.8}$$

and since $\gamma = (\nu'/\nu_{\rm L})^{1/2}$ we obtain

$$= \frac{2}{3} c\sigma_{\rm T} \frac{U_{\rm B}}{\nu_{\rm L}} n_0 \int_{\nu_{\rm L}}^{\infty} \left(\frac{\nu'}{\nu_{\rm L}}\right)^{(1-p)/2} \delta(\nu - \nu') d\nu' \tag{s1.9}$$

such that finally

$$P_\nu = \frac{2}{3} c\sigma_{\rm T} n_0 \frac{U_{\rm B}}{\nu_{\rm L}} \left(\frac{\nu}{\nu_{\rm L}}\right)^{-\frac{p-1}{2}} \tag{s1.10}$$

**Question 2:** *Synchrotron radiation: Supernova Remnants*
A supernova remnant has a mean density of 1 electron per cubic-centimetre and a volume of 2 cubic parsecs. The mean magnetic field in the cluster is $10^{-7}$ T. The remnant contains electrons with $\gamma$-values between 1 and $10^6$, which have a power-law energy distribution $n(\gamma)d\gamma = n_0\gamma^{-5}d\gamma$.

a) Determine $n_0$, using that $\int_{\gamma_{\rm min}}^{\gamma_{\rm max}} n(\gamma)d\gamma$ gives the total number of electrons in the remnant.

*Solution:* The total number of electrons in the supernova remnant is

$$N = \int n dV = nV = 1\,{\rm cm}^{-3} \cdot 1\,{\rm pc}^3 = 1\,{\rm cm}^{-3} \cdot (3 \times 10^{18}\,{\rm cm})^3 = 5.4 \times 10^{55} \tag{s2.1}$$

On the other hand,

$$N = n_0 \int_1^{10^5} \gamma^{-5} d\gamma = n_0 \left[-\frac{\gamma^{-4}}{4}\right]_1^{10^6} = n_0 \left(\frac{1}{4} - \frac{10^{-24}}{4}\right) = \frac{n_0}{4} \tag{s2.2}$$

to very high precision, such that $n_0 = 1.35 \times 10^{55}$.

b) Determine the power emitted at frequency $\nu$ by the remnant using Eq. (1.1).

*Solution:* The power emitted at frequency $\nu$ is

$$P_\nu = \frac{2}{3} c\sigma_{\rm T} n_0 \frac{U_{\rm B}}{\nu_{\rm L}} \left(\frac{\nu}{\nu_{\rm L}}\right)^{-\frac{p-1}{2}} \tag{s2.3}$$

where

$$c = 3 \times 10^8\,{\rm m\,s}^{-1} \tag{s2.4}$$
$$\sigma_{\rm T} = 6.65 \times 10^{-25}\,{\rm m}^2 \tag{s2.5}$$
$$n_0 = 1.35 \times 10^{55} \quad \text{(see above)} \tag{s2.6}$$
$$U_{\rm B} = \frac{B^2}{2\mu_0} = \frac{10^{-14}\,{\rm T}^2}{2 \cdot 1.26 \times 10^{-6}{\rm H\,m}^{-1}} = 4 \times 10^{-9}\,{\rm J\,m}^{-3} \tag{s2.7}$$
$$\nu_{\rm L} = \frac{qB}{2\pi m} = \frac{1.6 \times 10^{-19}\,{\rm C} \cdot 10^{-7}\,{\rm T}}{2 \cdot \pi \cdot 9 \times 10^{-31}\,{\rm kg}} = 2800\,{\rm s}^{-1} \tag{s2.8}$$
$$\tag{s2.9}$$

such that

$$P_\nu = 1.33 \times 10^{-16} \cdot 1.35 \times 10^{55} \cdot \frac{4 \times 10^{-9}}{2800} \left(\frac{1}{2800}\right)^{-(5-1)/2} \left(\frac{\nu}{1\,{\rm Hz}}\right)^{-(5-1)/2} \tag{s2.10}$$
$$= 2 \times 10^{34} {\rm J\,s}^{-1}\,{\rm Hz}^{-1} \left(\frac{\nu}{1\,{\rm Hz}}\right)^{-2} \tag{s2.11}$$

c) By integrating $P_\nu$ over $\nu$ determine the total synchrotron luminosity of the remnant.

*Solution:* The total luminosity is given by

$$L = \int_{\nu_{\min}}^{\nu_{\max}} P_\nu d\nu = 2 \times 10^{34}\,\mathrm{J\,s^{-1}} \int_{\nu_{\min}}^{\nu_{\max}} \left(\frac{\nu}{1\,\mathrm{Hz}}\right)^{-2} d\nu \tag{s2.12}$$

The limits for the integration need to be chosen that they cover the whole range of frequencies over which the electrons radiate. The upper limit is therefore $\nu_{\max} = \infty$. For the lower limit, we choose $\nu_{\min} = \nu_{\mathrm{L}}$, since this is the approximate frequency where the assumptions entering $P_\nu$ break down. Performing the integral then gives a total luminosity of $7.2 \times 10^{30}$ W for the remnant.

d) Compute the total energy content of the supernova remnant. Note that the energy content has two contributors: the energy in particles and the energy stored in the magnetic field. Compare this total energy content with the luminosity emitted by the remnant in synchrotron radiation. Estimate how long the remnant can radiate at its present level. Discuss.

*Solution:* With a volume of $V = 2.7 \times 10^{49}\,\mathrm{m}^3$ and the $U_B$ found above, the total energy stored in the magnetic field of the remnant is

$$E_{\mathrm{mag}} = V \cdot U_B = 2.2 \times 10^{47}\,\mathrm{J}. \tag{s2.13}$$

The energy stored in particles is

$$E_{\mathrm{particles}} = n_0 \int_1^{10^6} \gamma mc^2 \gamma^{-5} d\gamma = 1.35 \times 10^{55} \cdot 8 \times 10^{-14}\,\mathrm{J} \int_1^{10^6} \gamma^{-4} d\gamma = 3.6 \times 10^{41}\,\mathrm{J} \tag{s2.14}$$

The remnant can therefore radiate for $T = E_{\mathrm{particles}}/L = 5 \times 10^{10}\,s = 1600$ years. Supernova remnants powered by synchrotron radiations are therefore shortlived unless there is a source of energy in the centre of the remnant, such as a magnetised neutron star.

(remark: one contributor to the total energy that was ignored above is the kinetic energy in the bulk motion of the expanding supernova remnant. This kinetic energy is significantly larger than the above values and allows the remnant to be visible for up to several 100000 years, although in later phases the major radiation processes are not synchrotorn radiation).

The arguments in this question can be turned around to estimate the minimum $B$-field strength of a synchrotron source: the total energy content of the source better be such that it can produce its observed luminosity.

**Question 3:** *Synchrotrons on Earth*

One of the world's largest electron positron synchrotrons, the Large Electron-Positron Collider (LEP) at CERN in Geneva, Switzerland was essentially a ring of 4500 m radius containing electrons with a typical energy of 50 GeV (the LEP is currently being replaced with the Large Hadron Collider, LHC, which will provide energies of several TeV).

a) Estimate the magnetic field strength needed to keep the electrons on their circular path.

*Solution:* The Larmor radius is given by

$$R = \frac{\gamma v}{\Omega_{\mathrm{L}}} = \frac{\gamma v m}{eB} \tag{s3.1}$$

and therefore

$$B = \frac{\gamma v m}{eR} \tag{s3.2}$$

Since $E = \gamma mc^2 = 50\,\mathrm{GeV}$ and $mc^2 = 511\,\mathrm{keV} = 0.5\,\mathrm{MeV} = 5 \times 10^{-4}\,\mathrm{GeV}$, $\gamma = 10^5$, such that we can set $v = c$. Using $m = 9 \times 10^{-31}$ kg, and $e = 1.6 \times 10^{-19}$ C, we find $B = 0.0375$ T.

(in reality, much stronger magnets have to be used since the electrons are not kept on their path with such a simple $B$-field configuration as envisaged here).

b) Assuming that the energy loss of the electrons in the storage ring is solely due to synchrotron radiation, determine the power radiated by one electron in synchrotron radiation. In what waveband does the electron radiate most of its energy, i.e., what is its characteristic frequency and the corresponding photon energy?

*Solution:* As given in the lectures, the power emitted by the electron in synchrotron radiation is

$$P_{\mathrm{em}} = \frac{q^2}{6\pi c^3 \epsilon_0}\gamma^4 \frac{v^2 e^2 B^2}{\gamma^2 m^2} = 2.3 \times 10^{-7}\,\mathrm{J\,s^{-1}} \tag{s3.3}$$

using $v = c$, $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$ and the constants given above.

The characteristic frequency is

$$\nu_{\mathrm{c}} = \frac{\omega_{\mathrm{c}}}{2\pi} = \frac{eB\gamma^2}{m_{\mathrm{e}}c} = 2 \times 10^{11}\,\mathrm{Hz} \sim 200\,\mathrm{GHz} \tag{s3.4}$$

c) Determine the number of electrons in the storage ring, assuming that the electrons are responsible for a current of $I = 6\,\mathrm{mA}$.

*Solution:* By definition, the current in the storage ring is given by the total amount of charge flowing through the cross section $A$ of the storage ring per second:

$$I = \frac{Q}{T} \tag{s3.5}$$

The storage ring has a total volume of $V = 2\pi r A$ where $A$ is the (unknown) cross-section of the ring. Then the electron density in the ring is

$$n = N/V \tag{s3.6}$$

where $N$ is the total number of electrons in the ring.

Assuming that the electrons have a velocity $v = c$ (a good assumption, given their high energy), we can now compute how many electrons pass through $A$ in a time interval $T$. This number is equal to the number of electrons in a box of cross section $A$ and length $cT$. Therefore, the total charge passing through the cross section in a time $T$ is

$$Q = nAcTe \tag{s3.7}$$

where $e$ is the charge of the electron.

By definition, the current through $A$ is then given by

$$I = \frac{Q}{T} = \frac{nAcTe}{T} = \frac{NACe}{V} = \frac{Nce}{2\pi r} \tag{s3.8}$$

such that

$$N = \frac{2\pi r I}{ce} \sim 10^{12} \tag{s3.9}$$

(using $e = 1.6 \times 10^{-19}\,\mathrm{C}$ and $c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$).

d) Using these results, determine the power emitted in synchrotron radiation by the electrons.

*Solution:* The total power radiated by the synchrotron ring is

$$P = 2.3 \times 10^{-7}\,\mathrm{J\,s^{-1}} \cdot \times 10^{12} = 230\,\mathrm{kW} \tag{s3.10}$$

*Remark:* The numbers you will find, although already impressive, will grossly underestimate the total power requirements of a modern synchrotron (by a factor of about 60). The reason for this is that the major cause for accelerations of the electrons is not the acceleration to keep the electrons on a circular orbit, but small scale disturbances, e.g., in the $B$-field of the synchrotron, resulting in extremely large accelerations.

**Question 4:** *Comments on this week's lectures*

In order to improve the teaching and to enable myself to react to problems you might have with the module, I would like to hear your opinion on my teaching as early as possible. I would appreciate it if you would voice any problems and criticisms as soon as possible, e.g., on the speed with which I talk about the subjects of the lectures, the overall difficulty level of the class and the homework, the quality and contents of the handouts, and so on.

Please write these comments on a separate sheet of paper and give them to me: Either put the paper on the lectern before class or put it in my "pigeon hole" in the mailboxes on the 5th floor of the physics building, close to the physics undergraduate office. Feel free to remain anonymous, if you deem this necessary. You can also ask questions by sending email to `j.wilms@warwick.ac.uk`.

*Solutions to all questions can be found at `http://pulsar.astro.warwick.ac.uk/wilms/teach/astrospace/handouts.html`.*