Department of Physics
3^{rd} Year Modules 2004/2005
Term 1, Week 01–05

THE UNIVERSITY OF WARWICK

PX318: Astrophysics from Space – Example Solutions

Academic Week 04: Comptonization

Question 1: *Power law slope for Compton scattering*
NB: This question is rather lengthy, but given that it is currently rainy and windy, it is still well worth your time…
***NOTE:* The question as originally distributed during the lectures contained a few typos. Sorry!**

a) A slab of thickness ℓ and electron number density n (measured in units of electrons m^{-3}) is irradiated by light with initial intensity N_0 (where N is the number of photons per second and square-metre). Convince yourself that due to scattering, the decrease in photon number over infinitesimal distance dx is given by

$$\frac{dN}{N} = -n\sigma dx \tag{1.1}$$

where σ is the Thomson cross section. Use Eq. 1.1 to show that the number of photons emerging in the original direction of the photons on the other side of the slab is

$$N(\ell) = N_0 \exp(-\tau) \tag{1.2}$$

where $\tau = n\sigma\ell$.

Solution: For an infinitely thin slab, the area density of electrons is given by $n \cdot dx$ (units: electrons per square metre). For the purposes of scattering, each electron can be seen to have an area σ, such that the area $n\sigma dx$ is can be considered covered by the electrons. Once a photon hits this area, it is scattered out of the line of sight, and therefore the decrease in photon flux is

$$dN = -n\sigma N dx \tag{s1.1}$$

Writing this equation as

$$\frac{1}{N}\frac{dN}{dx} = -n\sigma \tag{s1.2}$$

and integrating over dx then gives

$$\int_0^\ell \frac{1}{N}\frac{dN}{dx}dx = -\int_0^\ell n\sigma dx \quad \longleftrightarrow \quad \log\left(\frac{N(\ell)}{N_0}\right) = -n\sigma\ell \tag{s1.3}$$

and therefore

$$N(\ell) = N_0 \exp(-n\sigma\ell) = N_0 \exp(-\tau) \tag{s1.4}$$

b) Using Eq. (1.2), convince yourself that the probability of a photon to travel at least an optical depth τ is

$$p(\tau) = \exp(-\tau) \tag{1.5}$$

and that the mean optical depth traveled before the photon scatters, $\langle\tau\rangle = 1$.

Solution: The mean optical depth traveled is given by

$$\langle\tau\rangle = \frac{\int_0^\infty \tau \exp(-\tau)d\tau}{\int_0^\infty \exp(-\tau)d\tau} \tag{s1.5}$$

The denominator integrates to 1 such that

$$\langle\tau\rangle = \int_0^\infty \tau \exp(-\tau) d\tau = [\exp(-\tau)\tau]_0^\infty + \int_0^\infty \exp(-\tau) d\tau = 1 \tag{s1.6}$$

where partial integration was used to solve the integral.

c) Use the result from the previous section to show that the mean physical distance traveled in the slab, the *mean free path l*, is

$$l = \frac{1}{n\sigma} \tag{1.7}$$

Solution: The mean optical depth traveled is $\langle\tau\rangle = n\sigma\ell = 1$, and therefore the mean distance traveled between scatters is

$$\ell = \frac{1}{n\sigma} \tag{s1.7}$$

d) Show that for small τ the probability of a photon undergoing k scatterings before escaping the medium is approximately

$$p_k(\tau) \sim \tau^k \tag{1.8}$$

Solution: The probability that the photon escapes after one scattering is

$$p_1(\tau) = 1 - \exp(-\tau) \sim \tau \tag{s1.8}$$

for τ small. The desired answer then follows by induction.

e) For Compton scattering and a seed photon energy $E_s \ll kT$, the amplification factor is

$$A \sim \frac{4kT}{mc^2} \tag{1.9}$$

Show that after k scatterings the energy of the seed photon, E_k, is approximately

$$E_k \sim E_s A^k \tag{1.10}$$

Solution: The energy of the photon after one scattering is

$$E_1 = E_s A \tag{s1.9}$$

and by induction, its energy after k scatterings is

$$E_k = E_{k-1} A = E_s A^k \tag{s1.10}$$

f) Using Eqs. 1.8 and 1.10, show that the emergent intensity at energy E_k is a power law

$$N(E_k) = N(E_s)\left(\frac{E_k}{E_s}\right)^{-\alpha} \quad \text{where} \quad \alpha = -\frac{\ln\tau}{\ln A} \tag{1.11}$$

Solution: The intensity emerging at energy E_k is approximately proportional to $p_k(\tau)$ since to first order only photons upscattered to E_k will contribute to the emerging spectrum. Therefore

$$N(E_k) = N(E_s) p_k(\tau) = N(E_s)\tau^k \tag{s1.11}$$

But because of $E_k = E_s A^k$,

$$k = \ln(E_k/E_s)/\ln A \tag{s1.12}$$

and therefore

$$\tau^k = \tau^{\ln(E_k/E_s)/\ln A} = (\exp(\ln\tau))^{\ln(E_k/E_s)/\ln A} = \exp\left(\ln(E_k/E_s)\ln\tau/\ln A\right) = \left(\frac{E_k}{E_s}\right)^{\ln\tau/\ln A} = \left(\frac{E_k}{E_s}\right)^{-\alpha} \tag{s1.13}$$

with $\alpha = -\ln\tau/\ln A$.

Question 2: *Even More Compton Scattering*

a) Show that the formula for the energy change of the electron,

$$E' = \frac{E}{1 + \frac{E}{m_e c^2}(1 - \cos\theta)} \tag{2.1}$$

can be written as

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) \tag{2.2}$$

where λ and λ' are the photon's wavelength before and after the scattering and where $h/m_e c$ is called the Compton wavelength.

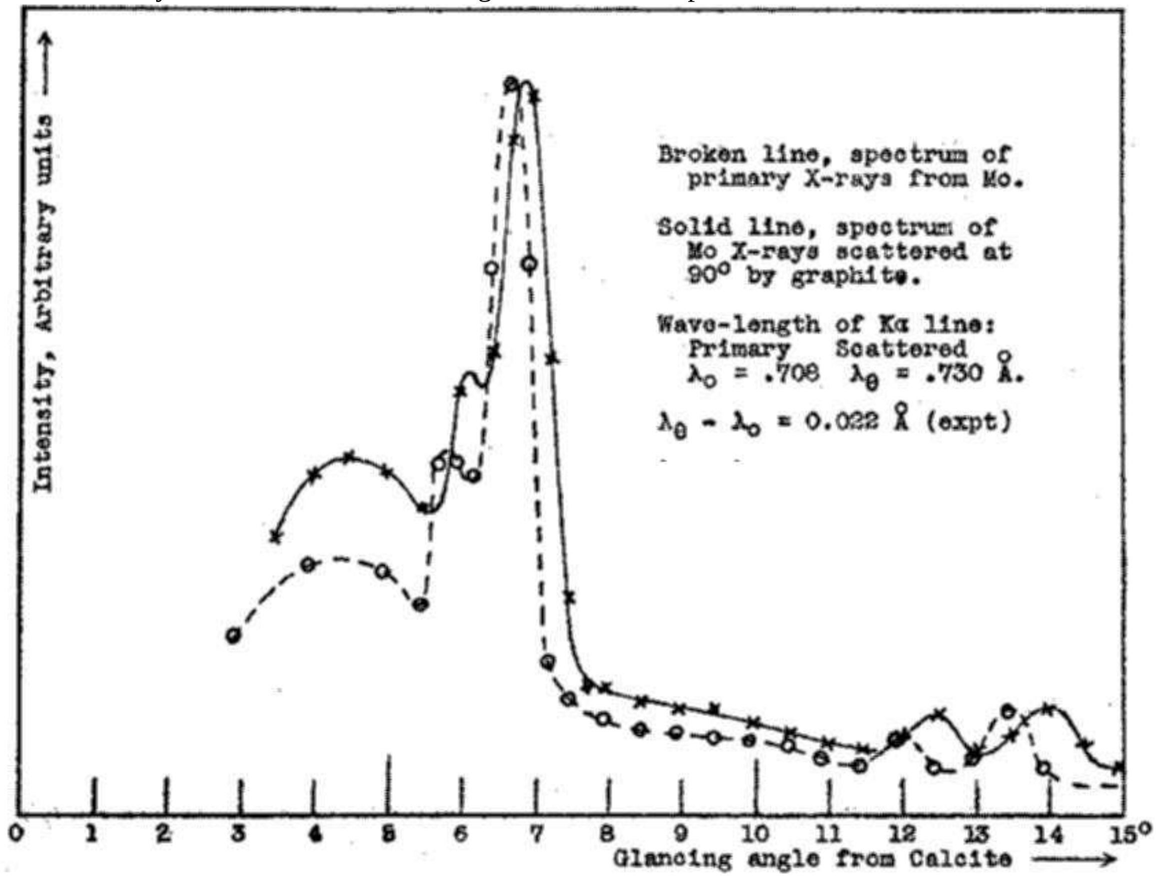
Solution: The photon energy is $E = h\nu = hc/\lambda$ and therefore

$$\frac{1}{\lambda'} = \frac{\frac{1}{\lambda}}{1 + \frac{1}{\lambda}\frac{h}{m_e c}(1 - \cos\theta)} \tag{s2.1}$$

such that

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\theta) \tag{s2.2}$$

b) The following figure is from A.H. Compton's discovery paper on the effect of electron scattering on photons, which eventually resulted in this effect being called the "Compton effect":



Compton irradiated a block of graphite with X-rays from a Molybdenium source. As shown in the figure, he found that the observed spectrum was shifted by 0.022 Å when looking at the graphite block under an angle of 90°. This small shift is due to Compton scattering in graphite (wavelength is measured in degrees in the figure above, since Compton used a crystal spectrometer to do his spectroscopy).

Compute the wavelength shift using the equations above and compare it to the measured value.

After doing this you might want to read Compton's original 1923 paper (Phys. Rev. 21, 483-502), available at http://prola.aps.org/abstract/PR/v21/i5/p483_1/p483 from all computers within the warwick.ac.uk domain. This paper is a beautiful piece of work, which finally convinced many physicists in the 1920s of the reality of the quantum nature of radiation. It is also one of the papers which eventually gained Compton his nobel prize in 1927, at age 35, "for the discovery of the effect named after him" (see http://nobelprize.org/physics/laureates/1927/compton-bio.html).

Solution: straightforward...

Question 3: *Synchrotron Self-Compton Radiation*
The broad-band spectra of radio-loud AGN are dominated by two humps: a low energy hump thought to be due to synchrotron radiation and a high energy hump, going up to TeV energies.

a) Compute the energy and γ-factor needed for an electron in a 10^{-7} T magnetic field to emit photons with an energy of 10 keV.

Solution: According to the lectures (Eq. 6.25), the characteristic radiation for synchrotron radiation of electrons with energy E in a magnetic field B is given by

$$\omega_\mathrm{c} = 2\pi\nu_\mathrm{c}\frac{eB}{m_\mathrm{e}}\left(\frac{E}{m_\mathrm{e}c^2}\right)^2 = \frac{eB}{m_\mathrm{e}}\gamma^2 \tag{s3.1}$$

Because of $E = h\nu = h\omega_\mathrm{c}/2\pi$, for a 10 keV photon:

$$\omega_\mathrm{c} = \frac{2\pi E}{h} = \frac{2\pi\cdot 10\,\mathrm{keV}}{4.136\times 10^{-18}\,\mathrm{keV\,s}} = 1.5\times 10^{19}\,\mathrm{s}^{-1} \tag{s3.2}$$

Therefore

$$\gamma = \sqrt{\frac{m_\mathrm{e}\omega_\mathrm{c}}{eB}} = \sqrt{\frac{9.11\times 10^{-31}\,\mathrm{kg}\cdot 1.5\times 10^{19}\,\mathrm{s}^{-1}}{1.6\times 10^{-19}\,\mathrm{C}\cdot 10^{-7}\,\mathrm{T}}} = 2.7\times 10^{8} \tag{s3.3}$$

b) What is the typical energy of such 10 keV photons after one Compton scattering with the electrons which produced them?

Solution: As shown in the lectures (Eq. 8.32), the approximate energy gain in relativistic Compton scattering is

$$E_\mathrm{after} = \gamma^2 E_\mathrm{before} = 7\times 10^{18}\,\mathrm{eV} \tag{s3.4}$$

Note: For a multitude of reasons, in realistic active galaxies, the γ-factors are significantly smaller than what you get in the answers to this question, however, the physics described here is in principle still the one powering the jets in active galaxies and quasars. In reality, the electrons in these systems make synchrotron radiation in the optical and UV, which is then upscattered into the gamma-rays, with correspondingly smaller γ-factors.

Question 4: *Comments on this week's lectures*
In order to improve the teaching and to enable myself to react to problems you might have with the module, I would like to hear your opinion on my teaching as early as possible. I would appreciate it if you would voice any problems and criticisms as soon as possible, e.g., on the speed with which I talk about the subjects of the lectures, the overall difficulty level of the class and the homework, the quality and contents of the handouts, and so on.
Please write these comments on a separate sheet of paper and give them to me: Either put the paper on the lectern before class or put it in my "pigeon hole" in the mailboxes on the 5th floor of the physics building, close to the physics undergraduate office. Feel free to remain anonymous, if you deem this necessary. You can also ask questions by sending email to `j.wilms@warwick.ac.uk`.

Solutions to all questions can be found at `http://pulsar.astro.warwick.ac.uk/wilms/teach/astrospace/handouts.html`.