Synchrotron Radiation

Introduction

To obtain the synchrotron radiation spectrum, we will have to perform the following steps:

1. Derive the motion of electrons in magnetic fields
2. Then use Larmor's formula to obtain the radiation characteristic from relativistic motion
3. Use the Doppler-effect to convert into the observer's frame of reference.
4. Integrate over electron distribution to obtain the final spectrum.

It is possible to do the same analytically without any approximations, however, we will use an approximate way here that is good enough to give the exact answer.

Synchrotron radiation (=Magnetobremsstrahlung) is the energy radiated by charged particles moving in magnetic fi elds. We already know how to compute the energy loss via Larmor's formula, so what remains to do is to compute the acceleration that a particle has in a magnetic fi eld.

We have to start, therefore, by looking at the motion of (relativistic) charges q in magnetic fi elds.

Assuming no electric fi eld is present (which is the case, given that the universe is charge neutral, i.e., we can assume the same number of electrons and protons present), this motion is described by the Lorentz-Force

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v}\times\mathbf{B} \quad \text{where} \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\beta^2}} = \gamma m\mathbf{v} \quad \text{and} \quad \beta = v/c. \tag{6.1}$$

We will assume that there are no radiative losses, i.e., the only force the charge feels is due to the magnetic fi eld. This is obviously not true on long timescales, because the particle is radiating synchrotron radiation, but it is generally true on short timescales, as we shall see later.

The velocity vector of the particle can be written in a component parallel to the magnetic fi eld, $\mathbf{v}_{\|}$, and in a component $\mathbf{v}_{\perp}$ perpendicular to the fi eld:

$$\mathbf{v}_{\|} = \frac{\mathbf{v}\cdot\mathbf{B}}{B}\frac{\mathbf{B}}{B} \qquad \mathbf{v}_{\perp} = \frac{\mathbf{B}\times(\mathbf{v}\times\mathbf{B})}{B^2} \tag{6.2}$$

$$|\mathbf{v}_{\|}| = v_{\|} = v\cos\alpha \qquad |\mathbf{v}_{\perp}| = v_{\perp} = v\sin\alpha \tag{6.3}$$

where α, the pitch-angle, is $\angle(\mathbf{v},\mathbf{B})$

Since

$$\mathbf{v}\times\mathbf{B} = (\mathbf{v}_{\|}+\mathbf{v}_{\perp})\times\mathbf{B} = \mathbf{v}_{\perp}\times\mathbf{B} \tag{6.4}$$

there is no acceleration parallel to the B-fi eld and we can ignore $v_{\|}$ for the moment. Since the force is perpendicular to $\mathbf{v}_{\perp}$ and $\mathbf{B}$ this means that we obtain the equations for circular motion around the magnetic fi eld:

$$ma_{\perp} = \frac{\gamma m v_{\perp}^2}{R} = qv_{\perp}B \tag{6.5}$$

such that

$$\frac{v_{\perp}}{R} = \frac{1}{\gamma}\frac{qB}{\gamma m} \tag{6.6}$$

The period to go around a circle at velocity v is

$$P = \frac{2\pi R}{v} \tag{6.7}$$

corresponding to a frequency

$$\omega = \frac{2\pi}{P} = \frac{v}{R} \tag{6.8}$$

Therefore we find the frequency with which the charged particle moves around the B-field is given by

$$\omega_B = \frac{1}{\gamma}\frac{qB}{m} = \frac{1}{\gamma}\omega_{\mathrm{L}} \tag{6.9}$$

where

$$\omega_{\mathrm{L}} = \frac{qB}{m} \quad \Longleftrightarrow \quad \nu_{\mathrm{L}} = \frac{\omega_{\mathrm{L}}}{2\pi} = \frac{qB}{2\pi m} \tag{6.10}$$

is called the *Larmor frequency* (also called the cyclo-frequency or the gyrofrequency.

The radius of the orbit of the particle, the *Larmor radius*, is

$$R = \frac{\gamma v_\perp}{\omega_{\mathrm{L}}} \tag{6.11}$$

To get a feel for some typical orders of magnitude, let's look at cosmic rays in the interstellar medium. These are produced, e.g., by acceleration of particles in supernova shock fronts. Instead of using the γ-factor of the particles as above, it is common to use their energy, E. Special relativity shows that the energy and γ are related by

$$E = \gamma mc^2 \tag{6.12}$$

Therefore, the Larmor radius is

$$R = \frac{\gamma vm}{qB} = \frac{\gamma mc^2}{qBc} = \frac{1}{qc}\frac{E}{B} \tag{6.13}$$

For cosmic rays, typical energies are $E = 1\,\mathrm{GeV} = 1.6 \times 10^{-10}\,\mathrm{J}$ and a typical magnetic field in our milky way is $B = 10^{-10}\,\mathrm{T}$. With these values we obtain $R = 3 \times 10^{11}\,\mathrm{m} \sim 2\,\mathrm{AU}$. This means that the radii are very small compared to typical length scales in the galaxy (which are measured in 10s of parsecs, where $1\,\mathrm{pc} = 206525\,\mathrm{AU}$). Therefore, we can say that cosmic rays are "frozen" into the magnetic field of our Galaxy and move "along" the magnetic field lines.

Radiated Energy

Electrodynamics: Radiation of an accelerated electron:

$$P_{\text{em}} = \frac{q^2}{6\pi c^3 \epsilon_0} \gamma^4 \left(\dot{v}_\perp^2 + \gamma^2 \dot{v}_\parallel^2 \right) \tag{6.14}$$

Derivation by Lorentz-Transforming the classical Larmor formula. Messy.

In case of circular motion, $\dot{v}_\perp = \omega_\text{B} \text{v}_\perp$. Hence

$$P_{\text{em}} = \frac{q^2}{6\pi c^3 \epsilon_0} \gamma^4 \frac{v_\perp^2 q^2 B^2}{\gamma^2 m^2} = 2\beta^2\gamma^2 c \cdot \sigma_\text{T} \cdot U_\text{B} \cdot \sin^2 \alpha \tag{6.15}$$

where (for electrons, i.e., $q = e$)

$$U_\text{B} = B^2/2\mu_0 \quad \text{(Energy density of the } B\text{-field),} \tag{6.16}$$

$$\sigma_\text{T} = \frac{\mu_0 e^4}{6\pi m_\text{e}^2 \epsilon_0 c^2} \quad \text{(Thomson-cross section)} \tag{6.17}$$

Presence of σ_T due to quantum electrodynamics: Derivation of synchrotron-radiation in frame of reference of electron via interaction of electron with a virtual photon of the magnetic field (i.e., Compton scattering with virtual photon).

Radiated Energy

Total energy radiated: Integration over all electrons.

Assumption: Isotropic velocity distribution.

Average pitch angle

$$\left\langle \sin^2\alpha \right\rangle = \frac{1}{4\pi}\int_0^{4\pi} \sin^2\alpha d\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin^2\alpha \sin\alpha \, d\alpha d\varphi = \frac{2}{3} \tag{6.18}$$

therefore

$$\langle P_{\text{em}}\rangle = \frac{4}{3}\beta^2\gamma^2 c\sigma_{\mathrm{T}} U_{\mathrm{B}} \tag{6.19}$$

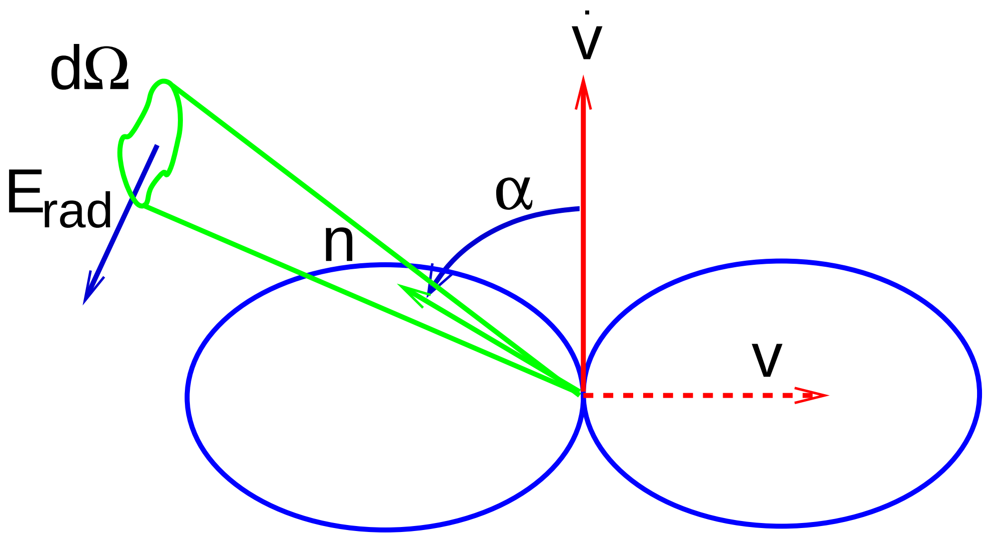
Note: Since $E = \gamma mc^2 \Longrightarrow P \propto E^2 U_{\mathrm{B}}$.

Note: $P_{\mathrm{em}} \propto \sigma_{\mathrm{T}} \propto m^{-2} \Longrightarrow$ Synchrotron radiation from charged particles with larger mass (protons,...) is negligible.

Note: Life-time of particle of energy E is

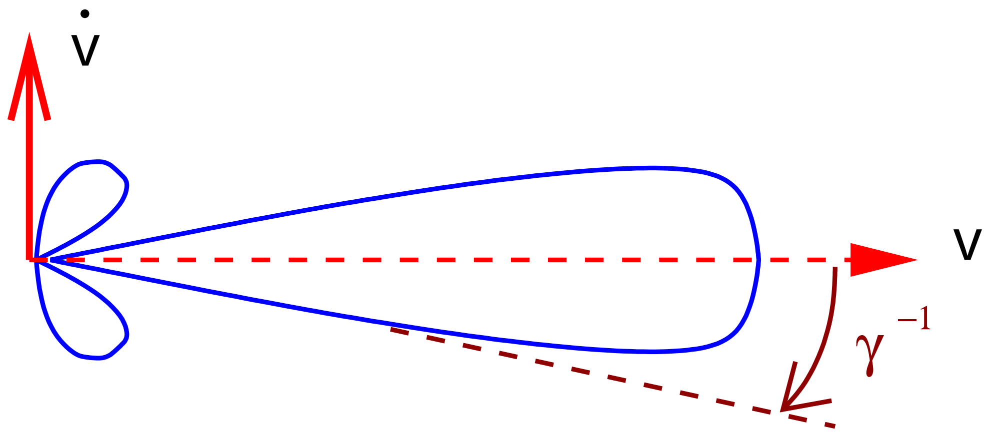
$$t_{1/2} \sim \frac{E}{P} \propto 1/(B^2E) = 5\,\text{s}\left(\frac{B}{1\,\text{T}}\right)^{-2}\gamma^{-1} = 1.6\times 10^7\,\text{years}\left(\frac{B}{10^{-7}\,\text{T}}\right)^{-2}\gamma^{-1} \tag{6.20}$$

Single Electron spectrum, I



after Rybicki & Lightman, Fig. 3.5

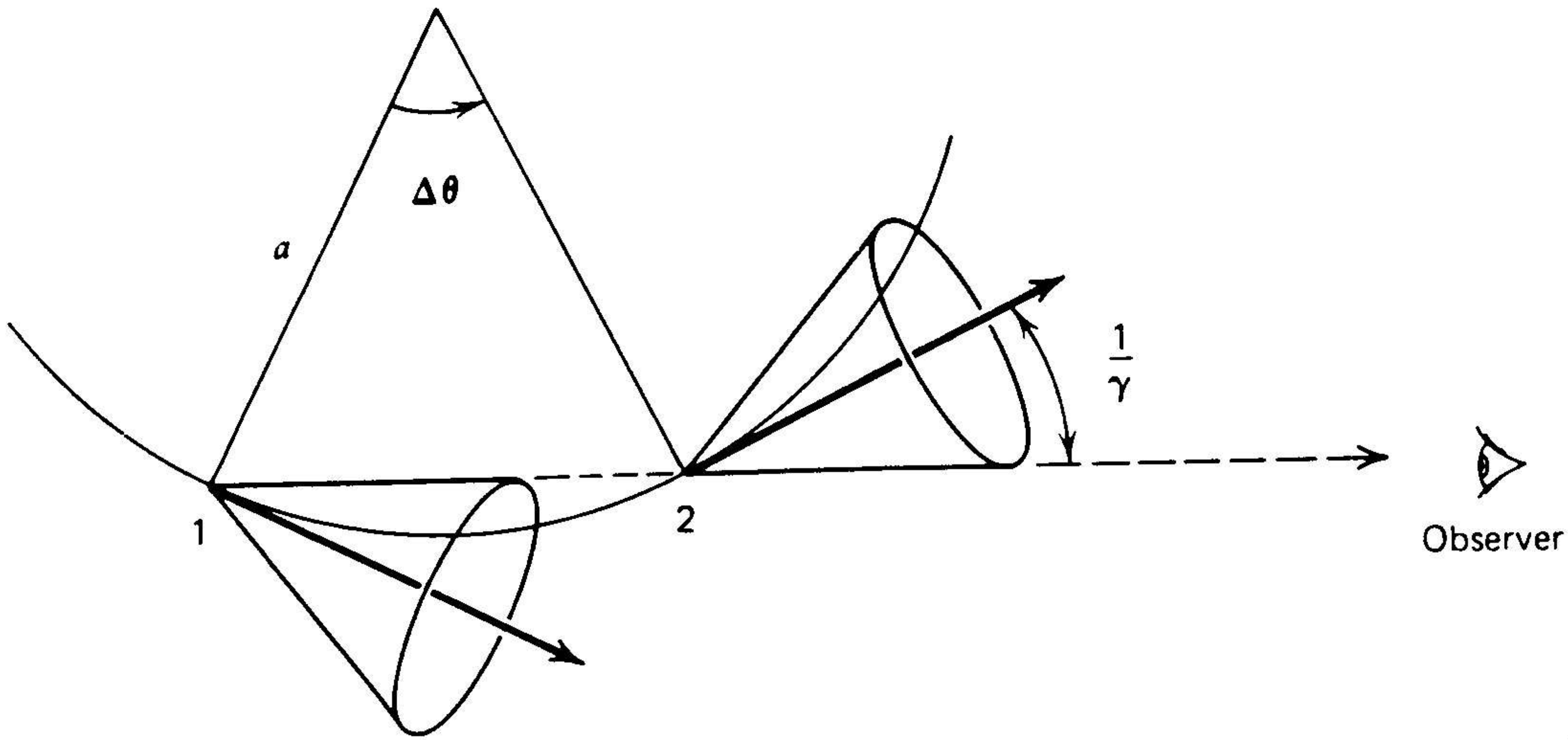
Frame of reference of electron: Emitted radiation has dipole characteristic (see, e.g., Eq. 5.7).



after Rybicki & Lightman, Fig. 4.11d

Lorentz-Transform into laboratory system: Forward Beaming. Opening angle is $\Delta\theta \approx \gamma^{-1}$.

Single Electron spectrum, II



Rybicki & Lightman, Fig. 6.2

In the electron frame of rest, beam passes observer during time

$$\Delta t = \frac{\Delta\theta}{\omega_{\rm B}} = \frac{m_{\rm e}\gamma}{eB}\frac{2}{\gamma} = \frac{2}{\omega_{\rm L}} \qquad (6.21)$$

Single Electron spectrum, III

But: Doppler effect shortens duration of pulse (electron is closer to observer at end of beam).

$\Longrightarrow$ Duration of pulse:

$$\tau = \left(1 - \frac{v}{c}\right)\Delta t = (1-\beta)\Delta t \tag{6.22}$$

For $\gamma \gg 1$, i.e., $\beta = v/c \sim 1$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = (1+\beta)(1-\beta) \sim 2(1-\beta) \tag{6.23}$$

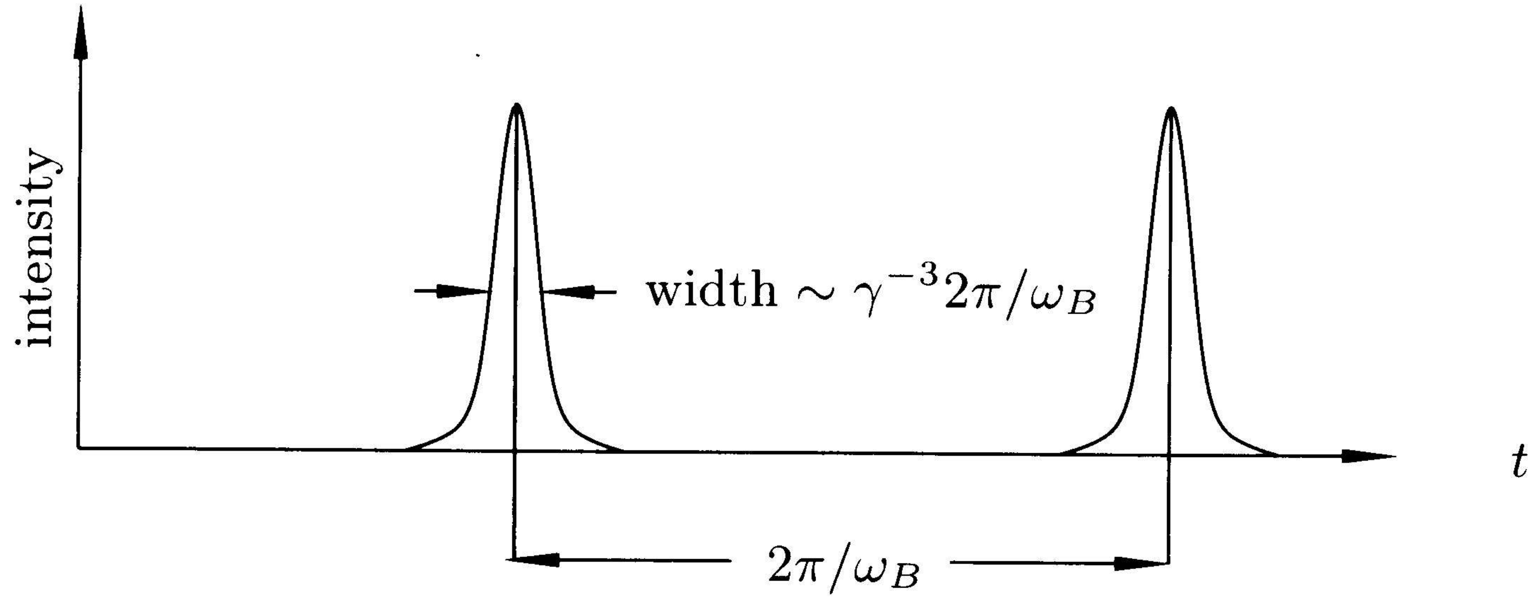
such that

$$\tau = \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)\Delta t = \frac{1}{\gamma^2\omega_{\mathrm{L}}} \tag{6.24}$$

Thus the characteristic frequency of the radiation is given by

$$\omega_{\mathrm{c}} = \gamma^2\omega_{\mathrm{L}} = \frac{eB}{m_{\mathrm{e}}}\left(\frac{E}{m_{\mathrm{e}}c^2}\right)^2 \tag{6.25}$$

Resulting Field



Shu, Fig. 18.2

The observed time-dependent E-Field, $E(t)$, from one electron is a sequence of pulses of width τ, separated by Δt.

Can approximate to good precision these single peaks by δ-functions.

In reality: Derive spectrum by Fourier-transforming $E(t)$. Basic result is the same.

Power-law distribution, I

Spectral energy distribution P_ν of an electron with total energy $E = \gamma m_\mathrm{e} c^2$ as

$$P_\nu(\gamma) = \frac{4}{3}\beta^2\gamma^2 c\sigma_\mathrm{T} U_\mathrm{B} \delta(\nu - \gamma^2\nu_\mathrm{L}) \tag{6.26}$$

where $\delta(x)$ is a δ-function, i.e.,

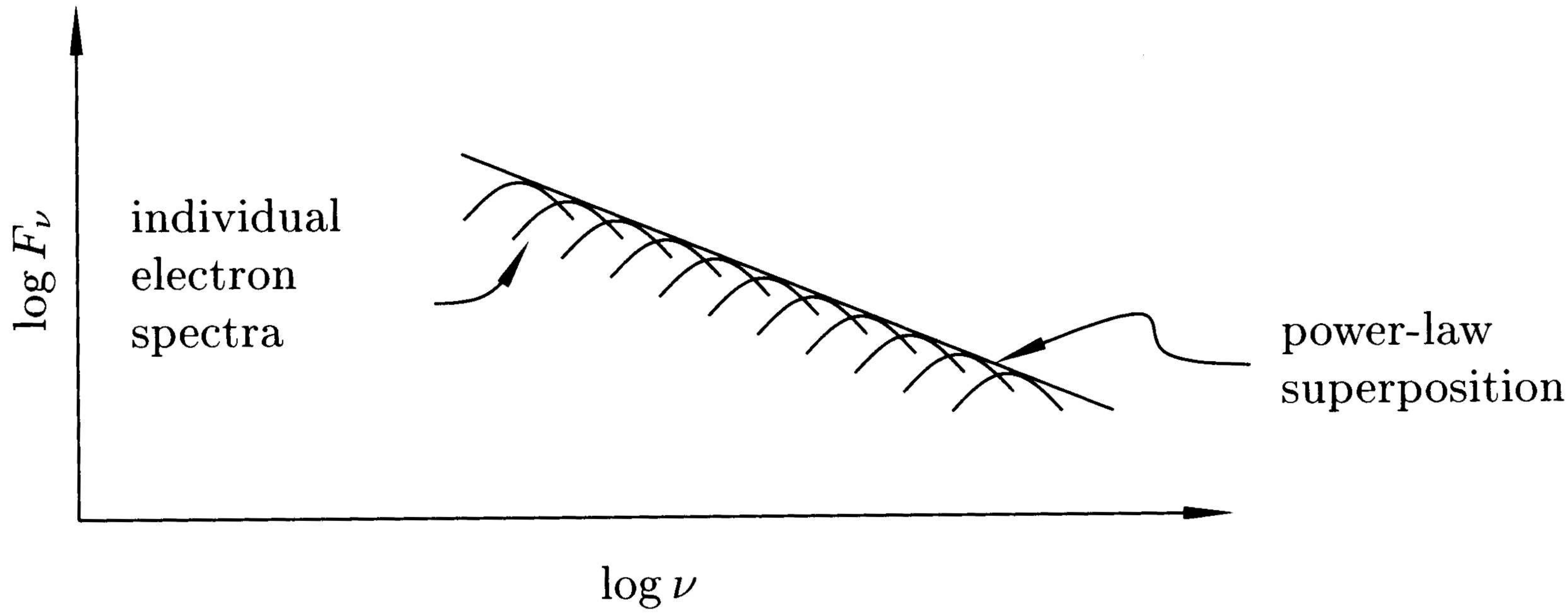
$$\delta(x) = 0 \quad \text{for } x \neq 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \tag{6.27}$$

i.e., electron with energy γmc^2 "blinks" at frequency $\nu = \gamma^2\nu_\mathrm{L}$.

By definition, for nonthermal synchrotron radiation, electrons have a power-law distribution

$$n(\gamma)d\gamma = n_0\gamma^{-p}d\gamma \tag{6.28}$$

Power-law distribution, II



Power emitted by electron distribution (=spectrum) found by integrating over all electrons

$$P_\nu = \int_1^\infty P_\nu(\gamma) n(\gamma) d\gamma \tag{6.29}$$

Power-law distribution, III

Therefore

$$P_\nu = \int_1^\infty \frac{4}{3}\beta^2\gamma^2 c\sigma_{\mathrm{T}} U_{\mathrm{B}} \delta(\nu - \gamma^2\nu_{\mathrm{L}}) n_0 \gamma^{-p} d\gamma \tag{6.30}$$

since $\gamma \gg 1$: $\beta \approx 1$

$$= A \int_1^\infty \gamma^{2-p} \delta(\nu - \gamma^2\nu_{\mathrm{L}}) d\gamma \tag{6.31}$$

substituting $\nu' = \gamma^2\nu_{\mathrm{L}}$, i.e., $d\nu' = 2\nu_{\mathrm{L}}\gamma d\gamma$

$$= B \int_{\nu_{\mathrm{L}}}^\infty \gamma^{1-p} \delta(\nu - \nu') d\nu' \tag{6.32}$$

since $\gamma = (\nu/\nu_{\mathrm{L}})^{1/2}$, one fi nally fi nds

$$P_\nu = \frac{2}{3} c\sigma_{\mathrm{T}} n_0 \frac{U_{\mathrm{B}}}{\nu_{\mathrm{L}}} \left(\frac{\nu}{\nu_{\mathrm{L}}}\right)^{-\frac{p-1}{2}} \tag{6.33}$$

The spectrum of an electron power-law distribution is a power-law!