# *Observational Cosmology*

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## *Introduction*



## Schedule

| | | | |
|---|---|---|---|
| Introduction | 01 | 16.10. | Introduction/History |
| | 02 | 23.10. | Basic Facts |
| World Models | 03 | 30.10. | World Models |
| Classical Cosmology | 04 | 06.11. | Distances, $H_0$ |
| | 05 | 13.11. | Distances, $H_0$ |
| The Early Universe | 06 | 20.11. | Hot Big Bang Model |
| | 07 | 27.11. | Nucleosynthesis |
| | 08 | 04.12. | Inflation |
| | | 11.12. | **no lecture** |
| Large Scale Structures | 09 | 18.12. | $\Omega$ and $\Lambda$ |
| | 10 | 08.01. | Dark Matter |
| | 11 | 15.01. | Large Scale Structures |
| | 12 | 22.01. | Structure Formation |
| | 13 | 29.01. | Structure Formation |
| Summary | 14 | 05.02. | Wrap Up |





## Literature

1. Cosmology Textbooks

SCHNEIDER, P., 2005, *Einführung in die Extragalaktische Astronomie und Kosmologie*, Heidelberg: Springer, 59.95€(English edition also available)

Well written introduction to cosmology, approximately at the level of this lecture. Recommended.

PEACOCK, J.A., 1999, *Cosmological Physics*, Cambridge: Cambridge Univ. Press, 49.50€

Very exhaustive, but difficult to read since the entropy per page is *very* high. . . still: a “must buy”.

LONGAIR, M.S., 1998, *Galaxy Formation*, Berlin: Springer, 53.45€

Clear and pedagogical treatment of structure formation, recommended.

## Literature

BERGSTRÖM, L. & GOOBAR, A., 1999, *Cosmology and Particle Astrophysics*, New York: Wiley, 47.90€

Nice description of the physics relevant to cosmology and high energy astrophysics, focusing on concepts. Less detailed than Peacock, but easier to digest.

PADMANABHAN, T., 1996, *Cosmology and Astrophysics Through Problems*, Cambridge: Cambridge Univ. Press, $36.95

Large collection of standard astrophysical problems (with solutions) ranging from radiation processes and hydrodynamics to cosmology and general relativity

PADMANABHAN, T., 1993, *Structure Formation in the Universe*, Cambridge: Cambridge Univ. Press, 46.50€

Mathematical treatment of cosmology, focusing on the formation of structure . . . Less astrophysical than the book by Longair.

ISLAM, J.N., 2002, *An Introduction to Mathematical Cosmology*, Cambridge: Cambridge Univ. Press, 42.50€

Useful summary of the facts of classical theoretical cosmology, recently revised.





## Literature

KOLB, E.W. & TURNER, M.S., 1990, *The Early Universe*, Reading: Addison-Wesley, 49.90€

Graduate-level text, the section on phase transitions and inflation in the early universe is especially recommended.

PEEBLES, P.J.E., 1993, *Principles of Physical Cosmology*, Princeton: Princeton Univ. Press (antiquarian only, do not pay more than $30!)

700p introduction to modern cosmology by one of its founders, in some parts quite readable, however, many forward references make the book very difficult to read for beginners.





## Literature

2. Textbooks on General Relativity

WEINBERG, S., 1972, *Gravitation and Cosmology*, New York: Wiley, 129€

Classical textbook on GR, still one of the best introductions. Nice section on classical cosmology.

SCHUTZ, B.F., 1985, *A First Course in General Relativity*, Cambridge: Cambridge Univ. Press, 45.90€

Nice and modern introduction to GR. The cosmology section is very short, though.

MISNER, C.W., THORNE, K.S. & WHEELER, J.A., 1973, *Gravitation*, San Francisco: Freeman, 104.90€

Commonly called “MTW”, this book is as heavy as the subject. . . Uses a weird notation. The cosmology section is outdated.

WALD, R.M., 1984, *General Relativity*, Chicago: Univ. Chicago Press (only antiquarian, ∼$40)

Modern introduction to GR for the mathematically inclined.





## *History*

## Prehistory


Pre-Babylonian astronomy: no written records known

But: Observations of the sky must have been important!

“Adorant” from the Geißenklösterle cave near Blaubeuren (Lkr. Ulm; 3.8 cm × 1.4 cm); Back side shows marks which have been interpreted as a lunar calendar.





## Babylon

Babylonian astronomy: Earliest astronomy with influence on us: ∼360 d year
⟹ sexagesimal system [360:60:60], 24h day, 12×30 d year,. . .

Enuma Elish myth (∼1100BC): Universe is place of battle between Earth and Sky, born from world parents.

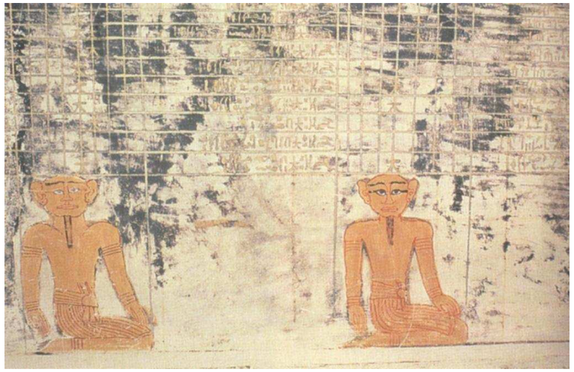
Note similar myth in the Genesis. . .

*Image:* Mul.Apin cuneiform tablet (British Museum, BM 86378, 8 cm high), describes rising and setting of constellations through the babylonian calendar. Summarizes astronomical knowledge as of before ∼690 BC.





## Egypt



Egyptian coffin lid showing two assistant astronomers, 2000. . . 1500 BC; hieroglyphs list stars (“decans”) whose rise defines the start of each hour of the night.

(Aveni, 1993, p. 42)

∼2000 BC: 365 d calendar (12×30 d plus 5 d extra), fixed to Nile flood (heliacal rising of Sirius), star clocks.

*heliacal rising:* first appearance of star in eastern sky at dawn, after it has been hidden by the Sun.





## Greek/Roman, I

Atlas Farnese, 2c A.D., Museo Archeologico Nazionale, Napoli

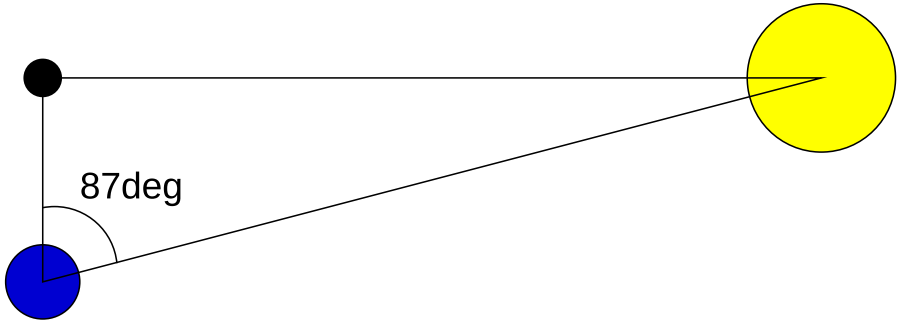
Early Greek astronomy: folk tale astronomy (Hesiod (730?–? BC), *Works and Days*). Constellations.

Thales (624–547 BC): Earth is flat, surrounded by water.

Anaxagoras (500–428 BC): Earth is flat, floats in nothingness, stars are far away, fixed on sphere rotating around us. Lunar eclipses: due to Earth’s shadow, Sun is hot iron sphere

Eudoxus (408–355 BC): Geocentric, planets affixed to concentric crystalline spheres. First real model for planetary motions!

## Greek/Roman, II




First attempts to measure scale of the universe:

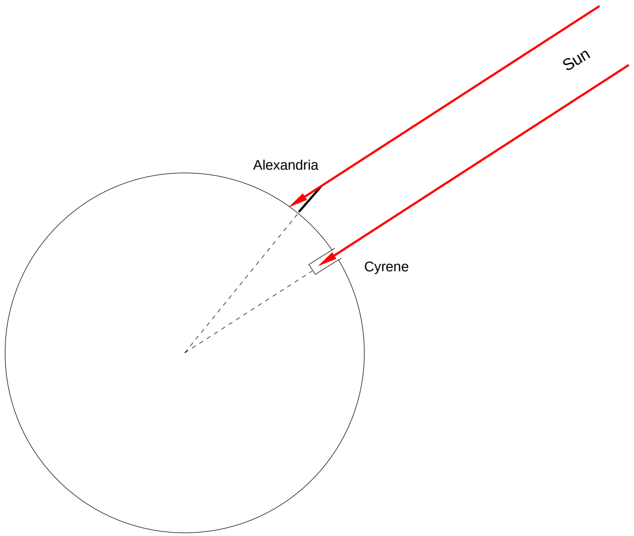
Aristarch (310–230 BC): Determination of the relative distance between the Moon and the Sun: Sun is 20$\times$ farther away than the Moon

reality: 400$\times$





## Greek/Roman, III



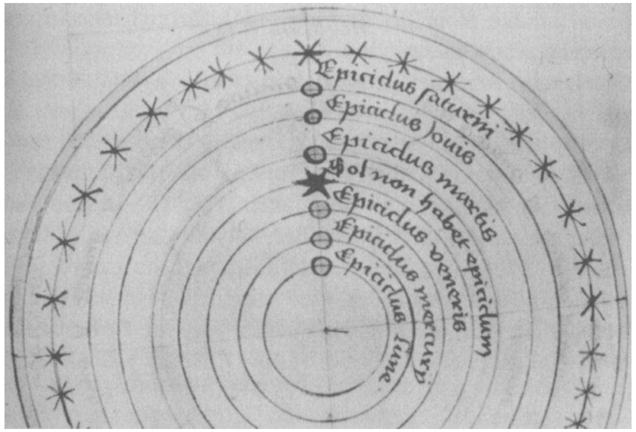

Eratosthenes von Cyrene (276–196 BC): Measurement of the radius of the Earth: Distance between Cyrene (Assuan) and Alexandria, diameter of Earth is 250000 stadia

The length of a stadium is unknown $\Longrightarrow$ we do not know how precise he was.





## Greek/Roman, IV



Aristotle (384–322 BC, *de caelo*): Refinement of Eudoxus model: add spheres to ensure smooth motion

$\Longrightarrow$ Universe filled with crystalline spheres (*nature abhors vacuum*).

Ether in celestial spheres, not on Earth (everything falls, except for planets and stars); Stars are very distant since they do not show parallaxes.

$\Longrightarrow$ Central philosophy until $\sim$1450AD!





## Hipparcus

Hipparchus (?? – $\sim$127 BC): Refinement of geocentric Aristotelian model into tool to make predictions.

- Catalogue of 850 stars
- magnitudes
- lunar parallax
- Table of “chords” (=early trigonometry)
- Discovery of precession

  Difference between the durations of the siderial and the tropical year [$365.25 - 1/300$ d vs. $365.25 + 1/400$ d], through comparison with babylonian measurements
- different duration of seasons
- conversion of geocentric model of Aristotele into a tool to make predictions.

## Ptolemy, I

Ptolemy (∼140AD): *Syntaxis* (aka Almagest): Refinement of Aristotelian theory into model useable for computations

Foundation of astronomy until Copernicus

⟹ Ptolemaic System.

(Aveni, 1993, p. 58)



After Hipparcus and Ptolemy: end of the golden age of early astronomy.

Greek works are continued by arabs and further refined.

Aristotele's philosophy remains foundation of science of medieval ages and is not questioned (in Europe).



## Renaissance

Nicolaus Copernicus (1473–1543): Earth centred Ptolemaic system is too complicated, a Sun-centred system is more elegant.





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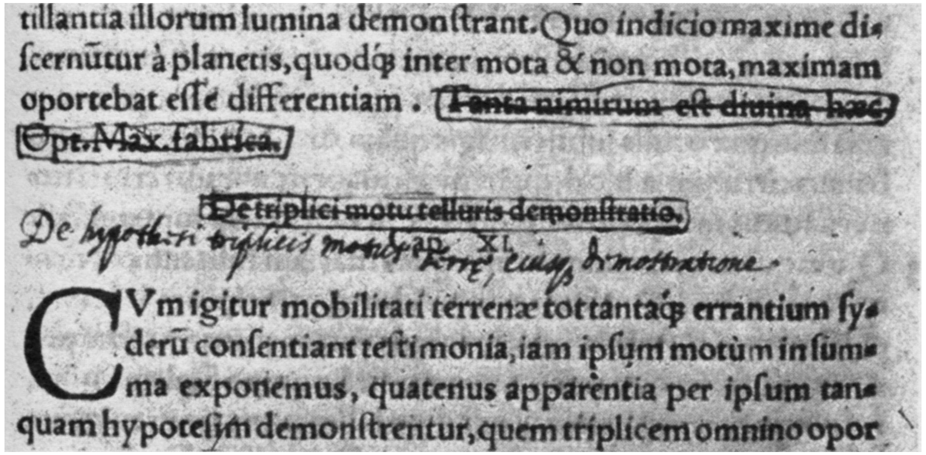
Copernican principle: The Earth is not at the center of the universe.





tillantia illorum lumina demonstrant. Quo indicio maxime di-
scernũtur à planetis, quodq́ inter mota & non mota, maximam
oportebat esse differentiam. ~~Tanta nimirum est diuina hæc~~
~~Opt. Max. fabrica.~~

~~De triplici motu telluris demonstratio.~~
De hypothesi triplicis motus terræ eiusq demonstratione Cap. XI.

CVm igitur mobilitati terrenæ tot tantaq́ errantium sy-
derũ consentiant testimonia, iam ipsum motũ in sum-
ma exponemus, quatenus apparentia per ipsum tan-
quam hypotesim demonstrentur, quem triplicem omnino opor

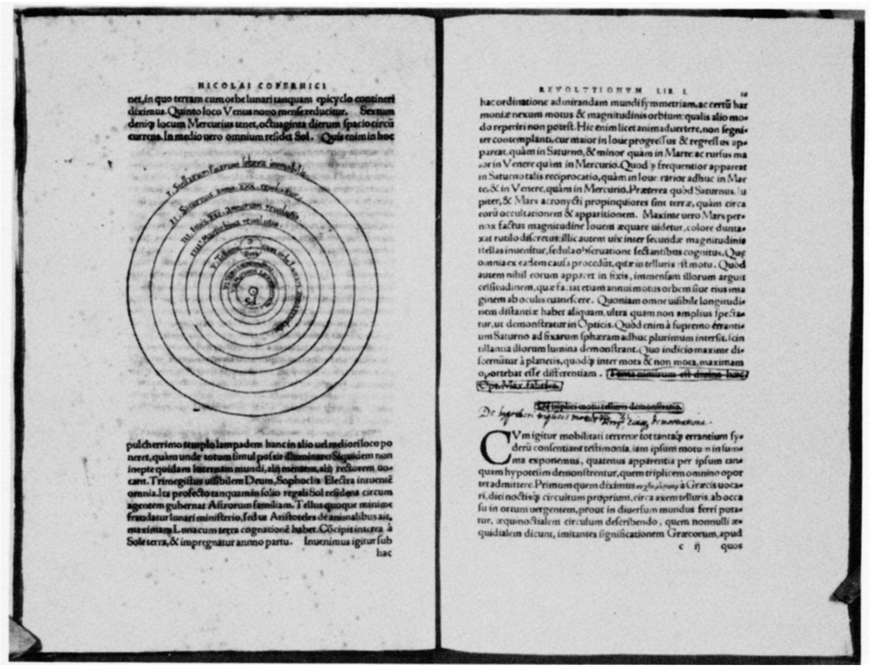
(Gingerich, 2005)

The “censored” copy of Galileo’s “de revolutionibus”

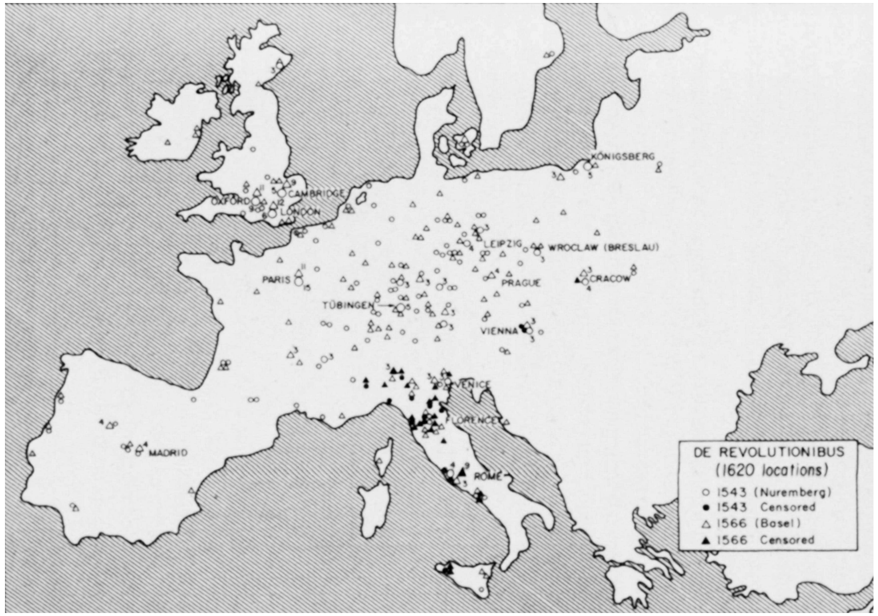
Deleted: “Indeed, large is the work of . . . God”

Changed: “On the explanation of the triple motion of the Earth”

$\Longrightarrow$ “On the hypothesis of the triple motion of the Earth”

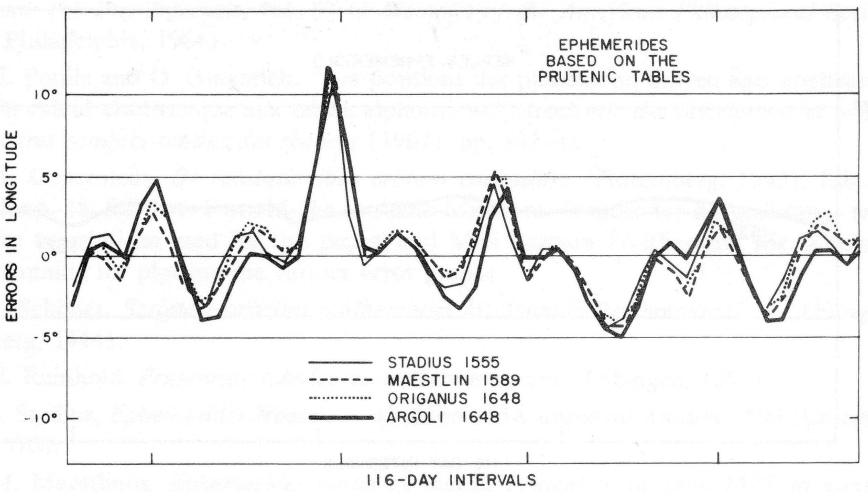


The “censored” copy owned by Galileo (Gingerich, 2005, Bibl. Florenz)




(Gingerich, 2005)

Distribution of the censored copies of “De revolutionibus”




(Gingerich, 1993)

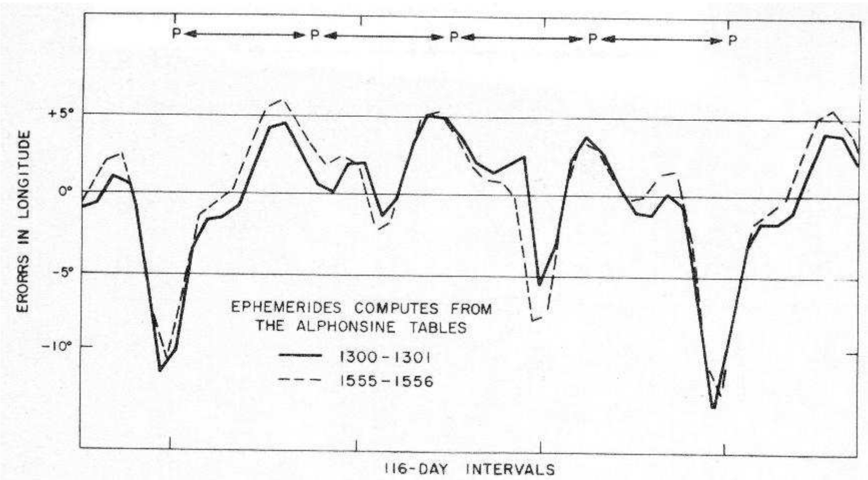
The error in the Copernican position of Mercury. . .



## Renaissance

Tycho Brahe (1546–1601): Visual planetary positions of highest precision reveal flaws in Ptolemaic positions.






. . . is not smaller than the error in the ptolemaic Alfonsinian Tables

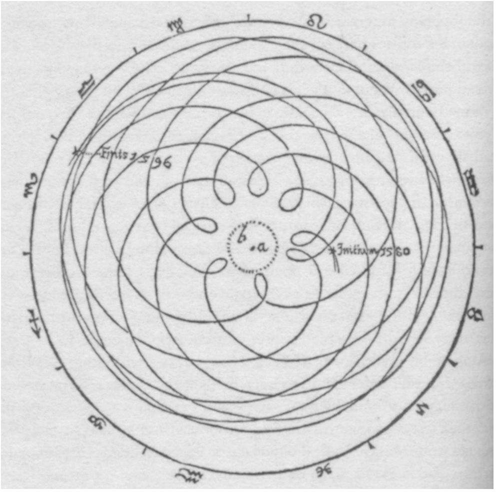


## Renaissance


Johannes Kepler (1571–1630):

- 27.12.1571, Weil der Stadt
- Studies in Tübingen with Maestlin
- 1594–1600: Graz
- 1596: Mysterium Cosmographicum
- 1600–1612: Prag, with Brahe, court astrologer, theory of planets, discovery of the supernova of 1604,. . .
- 1609: Astronomia Nova

## Renaissance



Kepler's theory of planetary motion: Astronomia nova (Prag, 1609)

Critique of epicycles: "panis quadragesimalis" (Osterbrezel)
$\Longrightarrow$ inelegant!

Astronomia Nova, chapter 1: Motion of Mars in the theory of epicycles



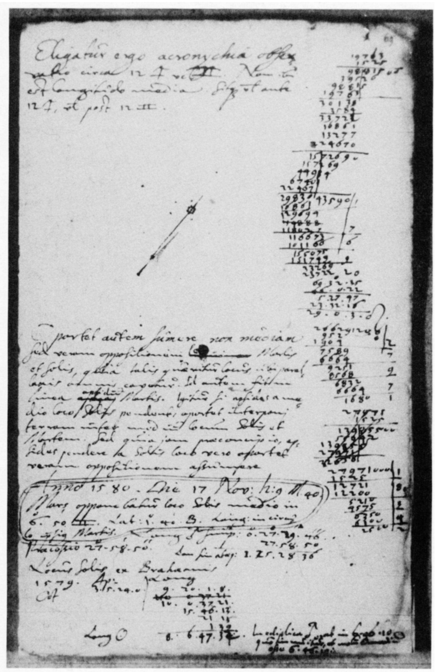


## Renaissance


Tabulae Rudolphinae, 1627
Best planetary positions
(error only $\sim 5'$!)

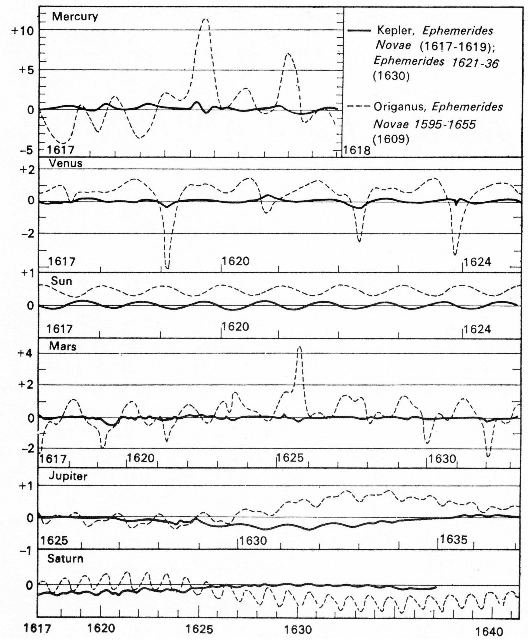
(Gingerich, 2005)





Kepler's laboratory book
Drawing of Mars in opposition highlighted: one of the few positions of Mars done by Brahe which Kepler was allowed to use

(Gingerich, 1993)




Comparison of positions, Kepler vs. copernican theory
$\Longrightarrow$ extreme improvement!

(Gingerich, 1993)

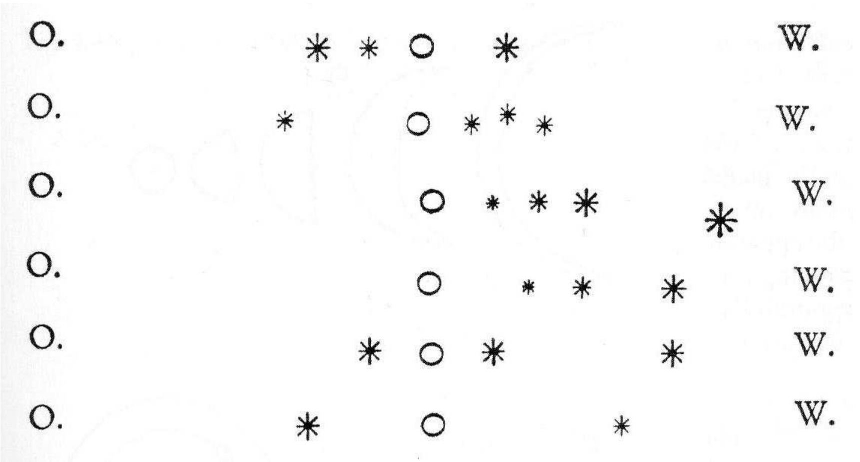
## Galileo Galilei, I

Galileo Galilei (1564–1642): Telescope
$\Longrightarrow$ Observations!
$\Longrightarrow$ Siderius Nuncius (1610)





## Galileo Galilei, II

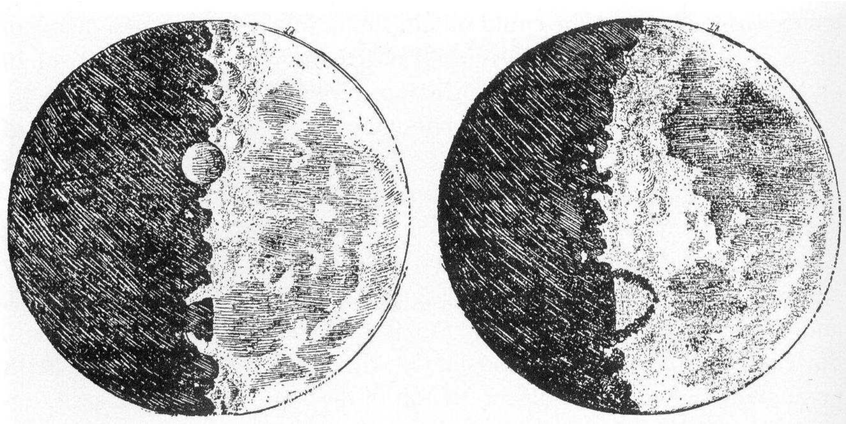



The moons of Jupiter move around Jupiter
($\Longrightarrow$ similar to the heliocentric model!)...





## Galileo Galilei, III

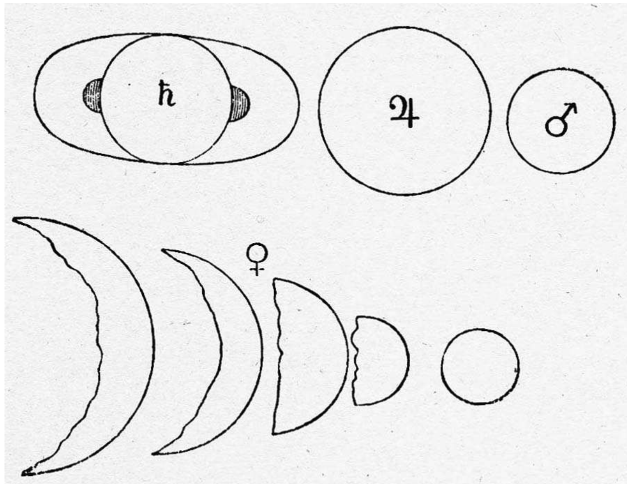


Moon has surface features, shadows, and “wiggles” (libration!).



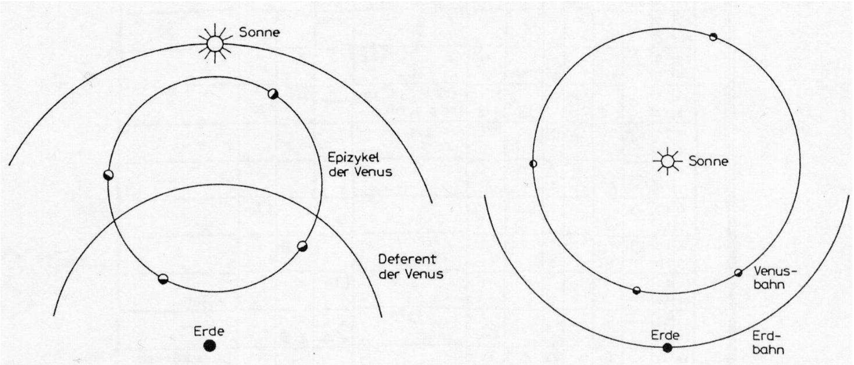


## Galileo Galilei, IV



Discovery of the phases of Venus (Il Saggiatore, 1623)

## Galileo Galilei, V




The observed sequence of the phases of Venus *cannot* be explained by the geocentric theory, only by a heliocentric theory.





## Newton

(Newton, 1730)

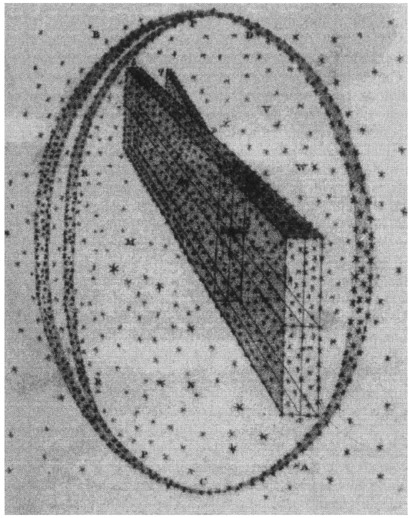
Isaac Newton (1642–1727): Newton's laws, physical cause for shape of orbits is gravitation
(*De Philosophiae Naturalis Principia Mathematica*, 1687).

$\Longrightarrow$ Begin of modern physics based astronomy.





## Modern Cosmology



Galileo: Milky Way consists of stars.

Newton: Stars are distant suns

William Herschel (1738–1822): Milky Way is a flattened disk of stars, Sun is at center (see figure).

Immanuel Kant (1724–1804): "Nebulae are galaxies" (disputed until the 1910s).

Friedrich Bessel (1784–1846): Distance to 61 Cyg (1838), positions of 50000 stars

John Herschel (1792–1871): General Catalogue of Galaxies (1864, 5079 Objects)

John Dreyer (1852–1926): NGC+IC (15000 Objects)

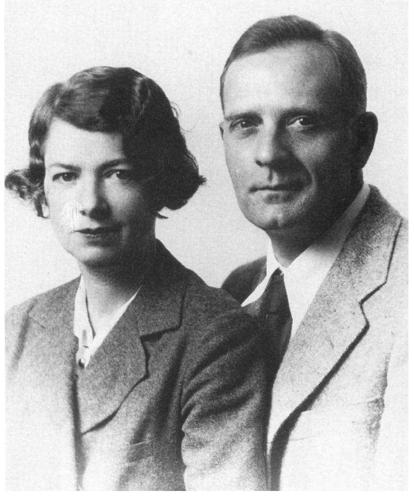




## Modern Cosmology

Albert Einstein (1879–1955): Theory of gravitation, applicability of theory to evolution of the universe as a whole.

## Edwin Hubble



Edwin Hubble (1889–1953):

- Realization of galaxies as being outside of the Milky Way
- Discovery that universe is expanding

Founder of modern extragalactic astronomy

Christianson, 1995, p. 165





Aveni, A. F., 1993, Ancient Astronomers, (Washington, D.C.: Smithsonian Books)

Gingerich, O., 1993, The Eye of Heaven – Ptolemy, Copernicus, Kepler, (New York: American Institute of Physics)

Gingerich, O., 2005, The book nobody read, (London: arrow books)

Newton, I., 1730, Opticks, Vol. 4th, (London: William Innys), reprint: Dover Publications, 1952



# *Basic Facts*



## Basic Facts

Cosmology deals with answering the questions about the universe as a whole.
The main question is:

How did the universe evolve into what it is now?

For this, *four major facts* need to be taken into account:

The universe is:
- expanding,
- isotropic,
- and homogeneous.

The isotropy and homogeneity of the universe is called the *cosmological principle*.
Perhaps (for us) the most important fact is:

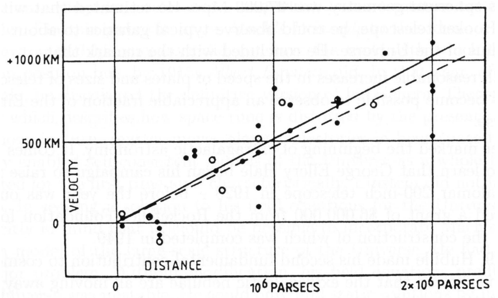
- The universe is habitable to humans.

i.e., the *anthropic principle*.

The one question cosmology **does not** attempt to answer is: How came the universe into being?

$\Longrightarrow$ Realm of theology!

## Expansion, I




(Hubble, 1929, Fig. 1)

Hubble (1929): Velocity $v$ (defined as $v/c := z = \Delta\lambda/\lambda$) for galaxy at distance $r$ is

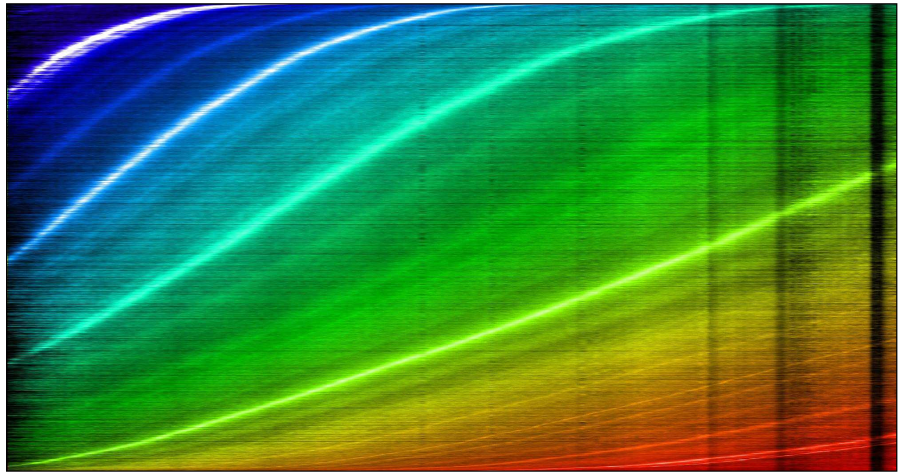
$$v(r) = H_0 r + v_X \cos\alpha\cos\delta + v_Y \sin\alpha\cos\delta + v_Z \sin\delta \quad (3.1)$$

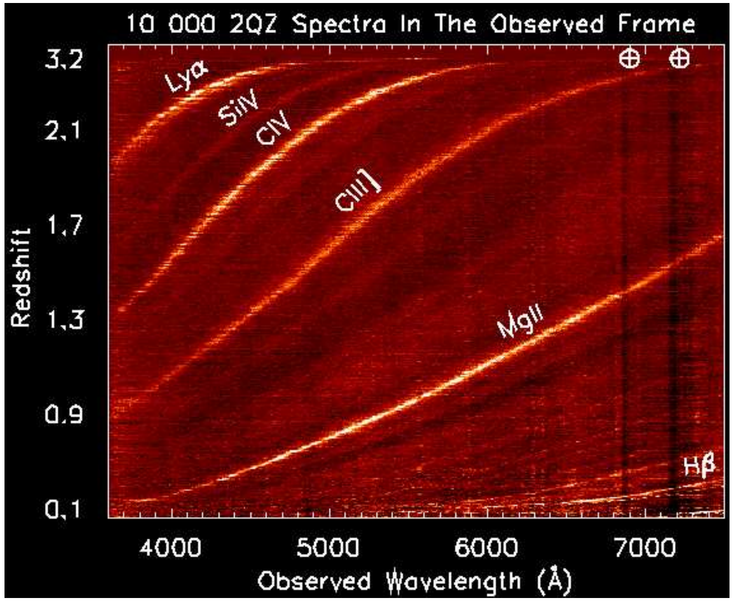
$(v_X, v_Y, v_Z)$ velocity due to motion of solar system ($\sim$ 350 km s$^{-1}$ towards $l = 264°$, $b = 48°$, Bennet et al., 1996)

$H_0$: "Hubble parameter"; *intrinsic* component of velocity due to *expansion* of the universe.

Old usage: "Hubble constant", but $H_0 \neq$ const. (cf. Eq. (4.37)).

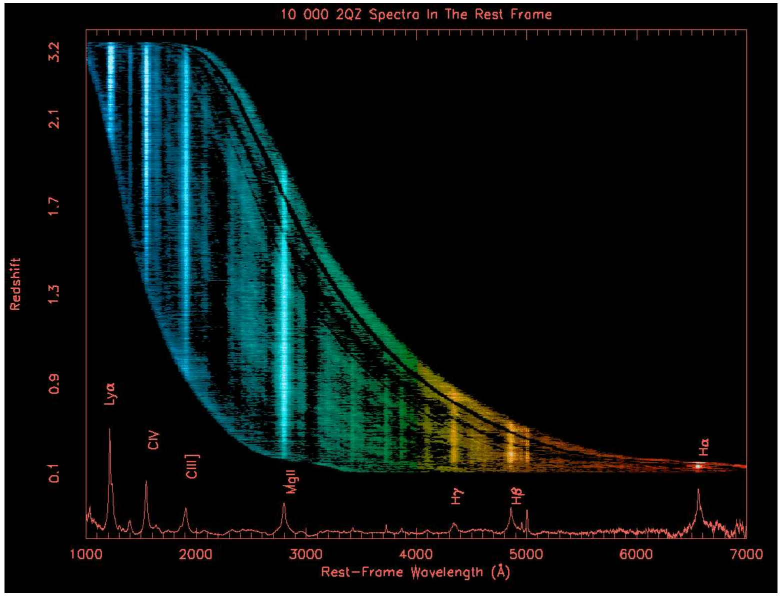




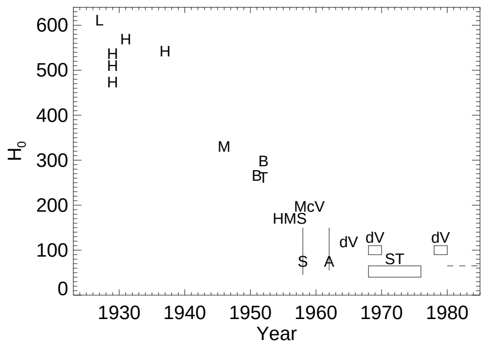




courtesy 2dF QSO Redshift survey

As a consequence of the cosmological redshift, for different $z$ different parts of the spectrum of a distant source are visible.



## Expansion, V




(after Trimble, 1997)

Currently accepted value:
$H_0 \sim 75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.
The systematic uncertainty of $H_0$ is $\sim 10\,\mathrm{km^{-1}\,s^{-1}\,Mpc^{-1}}$.
Parameterize uncertainty in formulae by defining

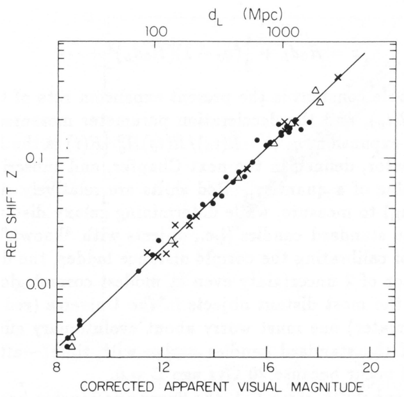
$$H_0 = 100\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \cdot h$$
$$H_0 = 75\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \cdot h_{75} \tag{3.2}$$

*Note:* $H_0^{-1}$ has units of time: $H_0^{-1} = 9.78\,\mathrm{Gyr}/h$: Hubble-Time;
for $h = 0.75$, the Hubble-Time is 13 Gyr.





## Expansion, VI




For standard candles, i.e., objects where the absolute luminosity $L$ is known, the Hubble law can be written using observed quantities only:
Euclidean space $\Longrightarrow$ observed flux

$$f = \frac{L}{4\pi d_\mathrm{L}^2} \iff d_\mathrm{L} = \left(\frac{L}{4\pi f}\right)^{1/2} \tag{3.3}$$

where $d_\mathrm{L}$ is the luminosity distance.
Using the Hubble law eq. (3.1)

$$H_0 d_\mathrm{L} = cz \implies z \propto H_0 \left(\frac{L}{4\pi f}\right)^{1/2} \tag{3.4}$$

Since *magnitudes* are defined via
$m \propto -2.5 \log f$:

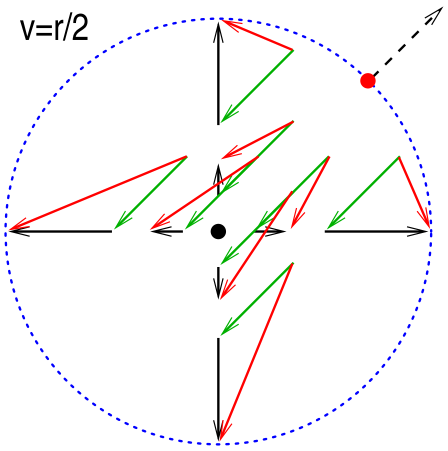
$$\log z \propto \log H_0 + \frac{1}{2}(\log L - \log f) \implies \log z = a + b(m - M) \tag{3.5}$$

where $m - M$: distance modulus.





## Expansion, VII




Expansion law $\boldsymbol{v} = H_0 \boldsymbol{r}$ is unchanged under rotation and translation: isomorphism.

*Proof:*

**Rotation:** Trivial.

**Translation:** Observations from place with position $\boldsymbol{r}'$ and velocity $\boldsymbol{v}'$: Observed distance is $\boldsymbol{r}_\mathrm{o} = \boldsymbol{r} - \boldsymbol{r}'$, observed velocity is $\boldsymbol{v}_\mathrm{o} = \boldsymbol{v} - \boldsymbol{v}'$. Because of the Hubble law,

$$\boldsymbol{v}_\mathrm{o} = H_0 \boldsymbol{r} - H_0 \boldsymbol{r}' = H_0 (\boldsymbol{r} - \boldsymbol{r}') = H_0 \boldsymbol{r}_\mathrm{o}$$

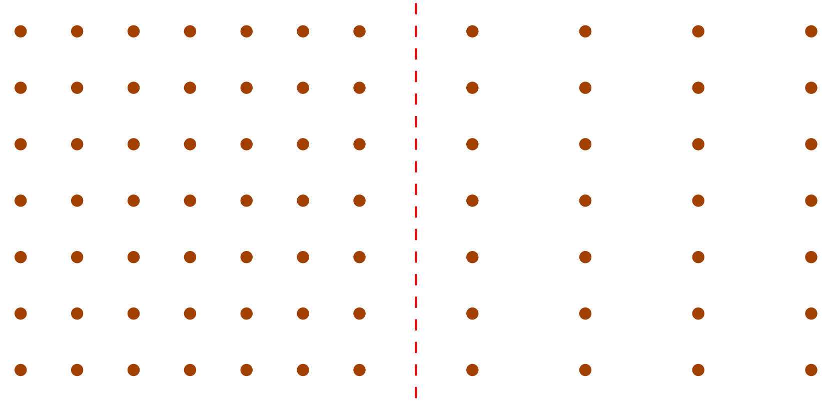
This isomorphism is a direct consequence of the homogeneity of the universe.

Despite everything receding from us, we are not at the center of the universe $\Longrightarrow$ Copernicus principle still holds.





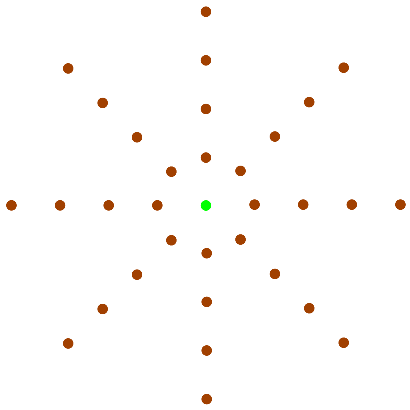
## Homogeneity and Isotropy, I



after Silk (1997, p. 8).

Note that homogeneity does not imply isotropy!

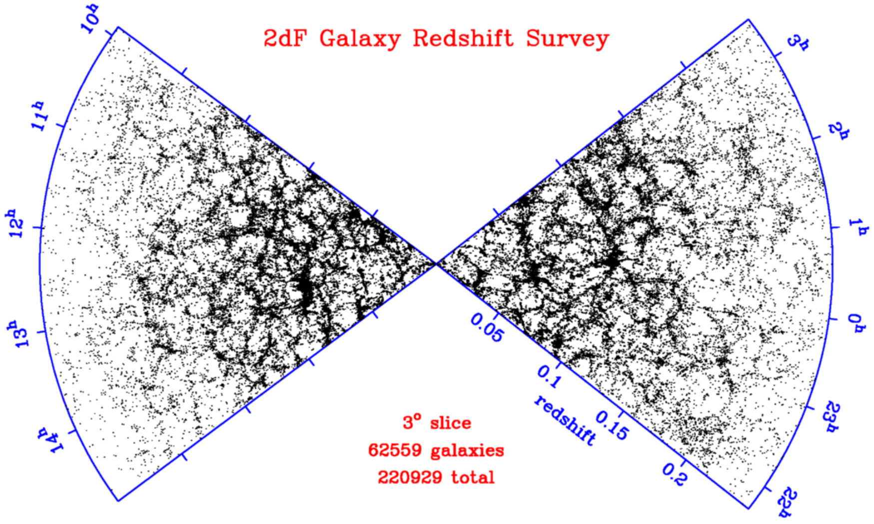
## Homogeneity and Isotropy, II



Neither does isotropy *around one point* imply homogeneity!

$\Longrightarrow$ *Both* assumptions need to be tested.






2dF Survey, $\sim$220000 galaxies total

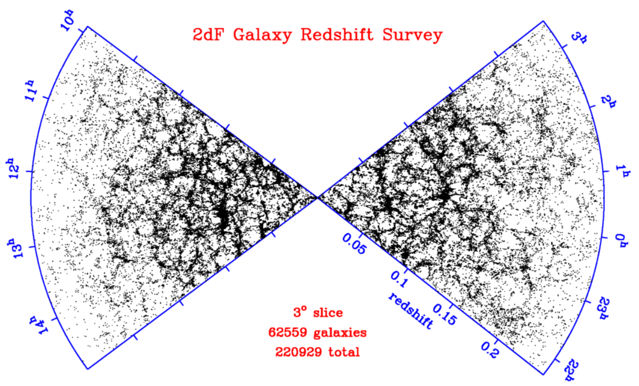
On scales $\gg$100 Mpc the universe looks indeed the same.

Below that: structure.

Structures seen are galaxy clusters (gravitationally bound) and superclusters (larger structures, not [yet] gravitationally bound).



## Homogeneity, I

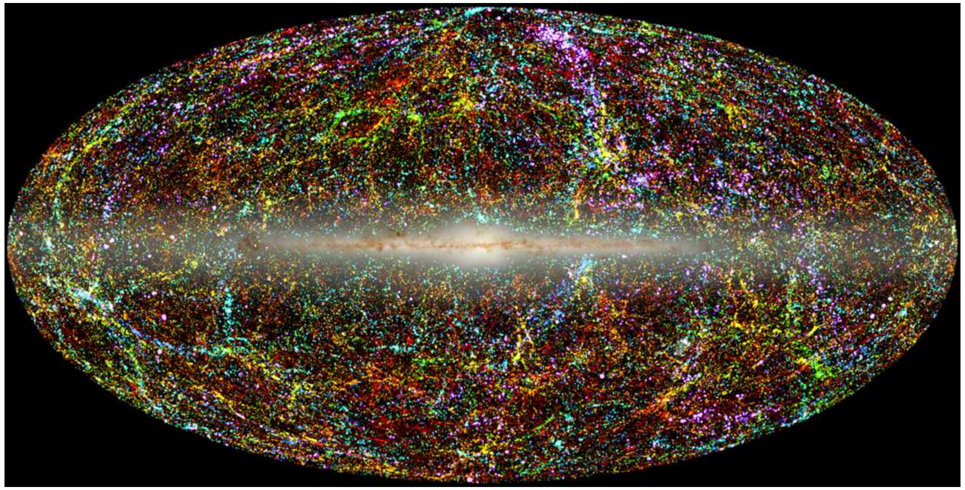



2dF Survey, $\sim$220000 galaxies total

The universe is homogeneous $\Longleftrightarrow$ The universe looks the same everywhere in space

Testable by observing spatial distribution of galaxies.

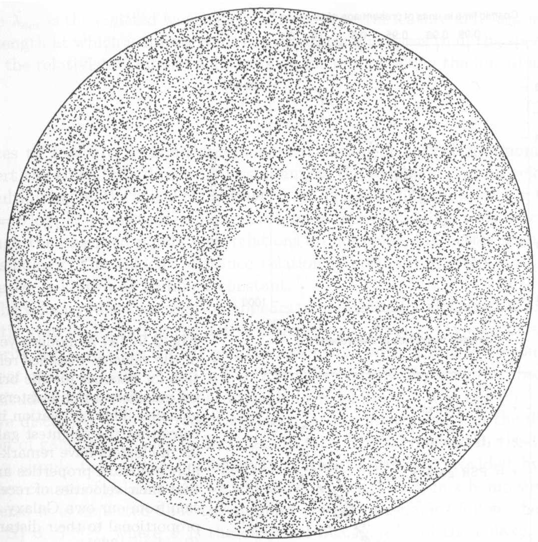




(Jarrett, 2004, Fig. 1)

Distribution of Galaxy redshifts in the 2MASS galaxy catalogue

## Isotropy



The universe is isotropic
$\Longleftrightarrow$ The universe looks the same in all directions

Radio galaxies are mainly quasars
$\Longrightarrow$ Sample large space volume ($z \gtrsim 1$)
$\Longrightarrow$ Clear isotropy.

Peebles (1993): Distribution of 31000 objects at $\lambda = 6$ cm from the Greenbank Catalogue.

Anisotropy in the image: galactic plane, exclusion region around Cyg A, Cas A, and the north celestial pole.






Bennet, C. L., et al., 1996, ApJ, 464, L1

Hubble, E. P., 1929, Proc. Natl. Acad. Sci. USA, 15, 168

Jarrett, T., 2004, Proc. Astron. Soc. Aust., 21, 396

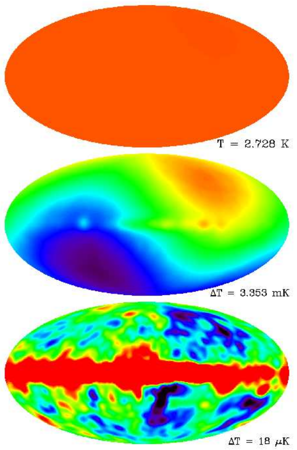
Peebles, P. J. E., 1993, Principles of Physical Cosmology, (Princeton: Princeton Univ. Press)

Silk, J., 1997, A Short History of the Universe, Scientific American Library 53, (New York: W. H. Freeman)

Trimble, V., 1997, Space Sci. Rev., 79, 793




## Isotropy




*Best evidence* for isotropy: Intensity of 3 K Cosmic Microwave Background (CMB) radiation.

First: dipole anisotropy due to motion of Sun (see slide 3–3), after subtraction: $\Delta T/T \lesssim 10^{-4}$ on scales from 10″ to 180°.

At level of $10^{-5}$: structure in CMB due to structure of surface of last scattering of the CMB photons, i.e., structure at the time when Hydrogen recombined.





# *World Models*

## Structure

Observations: cosmological principle holds: The universe is homogeneous and isotropic.

$\Longrightarrow$ Need theoretical framework obeying the cosmological principle.

Use combination of
- General Relativity
- Thermodynamics
- Quantum Mechanics

$\Longrightarrow$ Complicated!

For 99% of the work, the above points can be dealt with separately:
1. Define metric obeying cosmological principle.
2. Obtain equation for evolution of universe using Einstein field equations.
3. Use thermo/QM to obtain equation of state.
4. Solve equations.





## GRT vs. Newton

Before we can start to think about universe: Brief introduction to assumptions of general relativity.

$\Longrightarrow$ See theory lectures for the gory details, or check with the literature (Weinberg or MTW).

Assumptions of GRT:
- Space is 4-dimensional, might be curved
- Matter (=Energy) modifies space (Einstein field equation).
- Covariance: physical laws must be formulated in a coordinate-system independent way.
- Strong equivalence principle: There is no experiment by which one can distinguish between free falling coordinate systems and inertial systems.
- At each point, space is locally Minkowski (i.e., locally, SRT holds).

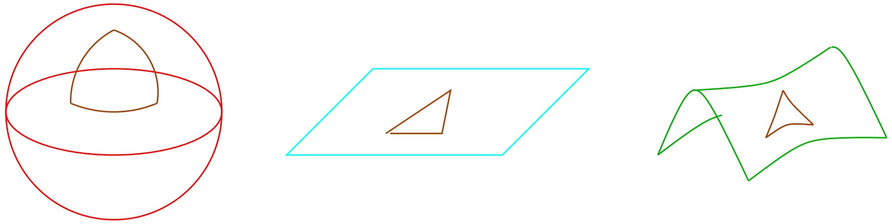
$\Longrightarrow$ Understanding of geometry of space necessary to understand physics.





## 2D Metrics

Before describing the 4D geometry of the universe: first look at 2D spaces (easier to visualize).



After Silk (1997, p. 107)

There are three classes of isotropic and homogeneous two-dimensional spaces:
- 2-sphere ($\mathscr{S}^2$) positively curved
- $x$-$y$-plane ($\mathbb{R}^2$) zero curvature
- hyperbolic plane ($\mathscr{H}^2$) negatively curved

(curvature $\approx \sum$angles in triangle $>$, $=$, or $<$ 180°)

We will now calculate what the metric for these spaces looks like.





## 2D Metrics

**The metric describes the local geometry of a space.**

Differential distance, $\mathrm{d}s$, in Euclidean space, $\mathbb{R}^2$:

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 \tag{4.1}$$

The metric tensor, $g_{\mu\nu}$, is defined through

$$\mathrm{d}s^2 = \sum_\mu \sum_\nu g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu =: g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{4.2}$$

(Einstein's summation convention)

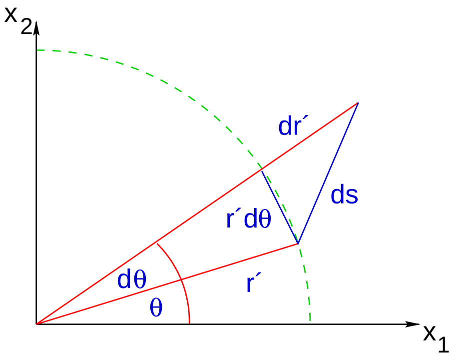
Thus, for the $\mathbb{R}^2$,

$$\begin{aligned} g_{11} &= 1 & g_{12} &= 0 \\ g_{21} &= 0 & g_{22} &= 1 \end{aligned} \tag{4.3}$$

## 2D Metrics

But: Other coordinate-systems are also possible in the plane!

Changing to polar coordinates $r'$, $\theta$, defined by




$$x_1 =: r' \cos\theta \qquad x_2 =: r' \sin\theta \tag{4.4}$$

it is easy to see that

$$ds^2 = dr'^2 + r'^2\, d\theta^2 \tag{4.5}$$

Performing a change of scale by substituting $r' = Rr$, then gives

$$ds^2 = R\{dr^2 + r^2\, d\theta^2\} \tag{4.6}$$

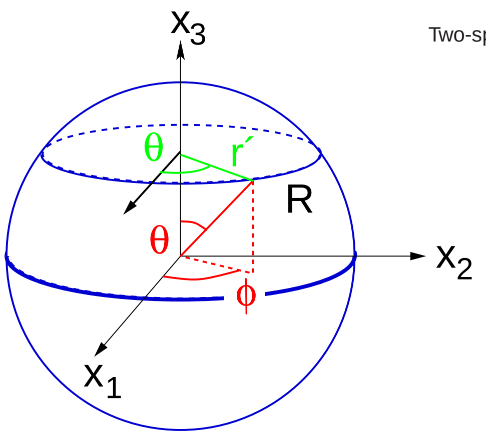




## 2D Metrics

A more complicated case occurs if space is curved.

*Easiest case:* surface of three-dimensional sphere (a two-sphere).




*After Kolb & Turner (1990, Fig. 2.1)*

Two-sphere with radius $R$ in $\mathbb{R}^3$:

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{4.7}$$

Length element of $\mathbb{R}^3$:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

Eq. (4.7) gives

$$x_3 = \sqrt{R^2 - x_1^2 - x_2^2}$$

such that

$$dx_3 = \frac{\partial x_3}{\partial x_1}\, dx_1 + \frac{\partial x_3}{\partial x_2}\, dx_2 = -\frac{x_1\, dx_1 + x_2\, dx_2}{\sqrt{R^2 - x_1^2 - x_2^2}} \tag{4.8}$$





## 2D Metrics

Introduce again polar coordinates $r'$, $\theta$ in $x_3$-plane:

$$x_1 =: r' \cos\theta x_2 \; =: r' \sin\theta \tag{4.4}$$

(note: $r'$, $\theta$ are only unique in upper or lower half-sphere)

The differentials are given by

$$dx_1 = \cos\theta\, dr' - r' \sin\theta\, d\theta \quad \text{and} \quad dx_2 = \sin\theta\, dr' + r'\cos\theta\, d\theta \tag{4.9}$$

In cartesian coordinates, the length element on $\mathscr{S}^2$ is

$$ds^2 = dx_1^2 + dx_2^2 + \frac{(x_1\, dx_1 + x_2\, dx_2)^2}{R^2 - x_1^2 - x_2^2} \tag{4.10}$$

inserting eq. (4.9) gives after some algebra

$$= r'^2\, d\theta^2 + \frac{R^2}{R^2 - r'^2}\, dr'^2 \tag{4.11}$$

finally, defining $r = r'/R$ (i.e., $0 \le r \le 1$) results in

$$ds^2 = R^2 \left\{ \frac{dr^2}{1 - r^2} + r^2\, d\theta^2 \right\} \tag{4.12}$$





## 2D Metrics

Alternatively, we can work in spherical coordinates on $\mathscr{S}^2$

$$x_1 = R \sin\theta \cos\phi \qquad x_2 = R\sin\theta\sin\phi \qquad x_3 = R\cos\theta \tag{4.13}$$

($\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$).

Going through the same steps as before, we obtain after some tedious algebra

$$ds^2 = R^2 \left\{ d\theta^2 + \sin^2\theta\, d\phi^2 \right\} \tag{4.14}$$

## 2D Metrics

(Important) remarks:

1. The 2-sphere has no **edges**, has no **boundaries**, but has still a **finite volume**, $V = 4\pi R^2$.
2. Expansion or contraction of sphere caused by **variation of** $R \Longrightarrow R$ determines the *scale* of volumes and distances on $\mathscr{S}^2$.

   $R$ is called the *scale factor*

3. Positions on $\mathscr{S}^2$ are defined, e.g., by $r$ and $\theta$, *independent* on the value of $R$

   $r$ and $\theta$ are called *comoving coordinates*

4. Although the **metrics** Eq. (4.10), (4.12), and (4.14) **look** *very* **different**, they **still describe the** *same* **space** $\Longrightarrow$ that's why physics should be covariant, i.e., independent of the coordinate system!





## 2D Metrics

The **hyperbolic plane**, $\mathscr{H}^2$, is defined by

$$x_1^2 + x_2^2 - x_3^2 = -R^2 \tag{4.15}$$

If we work in **Minkowski space**, where

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2 \tag{4.16}$$

then

$$= \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \frac{(x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2)^2}{R^2 + x_1^2 + x_2^2} \tag{4.17}$$

$\Longrightarrow$ substitute $R \to iR$ (where $i = \sqrt{-1}$) to obtain same form as for sphere (eq. 4.11)!

Therefore,

$$\mathrm{d}s^2 = R^2 \left\{ \frac{\mathrm{d}r^2}{1 + r^2} + r^2\,\mathrm{d}\theta^2 \right\} \tag{4.18}$$





## 2D Metrics

The analogy to spherical coordinates on the hyperbolic plane are given by

$$\begin{aligned} x_1 &= R \sinh\theta \cos\phi \\ x_2 &= R \sinh\theta \sin\phi \\ x_3 &= R \cosh\theta \end{aligned} \tag{4.19}$$

($\theta \in [-\infty, +\infty]$, $\phi \in [0, 2\pi]$).

A session with Maple (see handout) will convince you that these coordinates give

$$\mathrm{d}s^2 = R^2 \left\{ \mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2 \right\} \tag{4.20}$$

*Remark:*

$\mathscr{H}^2$ is **unbound** and has an **infinite volume**.





Transcript of Maple session to obtain Eq. (4.20):

```
> x1:=r*sinh(theta)*cos(phi);
```

$$x1 := r \sinh(\theta)\cos(\phi)$$

```
> x2:=r*sinh(theta)*sin(phi);
```

$$x2 := r \sinh(\theta)\sin(\phi)$$

```
> x3:=r*cosh(theta);
```

$$x3 := r \cosh(\theta)$$

```
> dx1:=diff(x1,theta)*dtheta+diff(x1,phi)*dphi;
```

$$dx1 := r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi$$

```
> dx2:=diff(x2,theta)*dtheta+diff(x2,phi)*dphi;
```

$$dx2 := r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi$$

```
> ds2:=dx1*dx1+dx2*dx2-(x1*dx1+x2*dx2)^2/(r^2+x 1^2+x2^2);
```

$$\begin{aligned} ds2 := &\,(r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi)^2 \\ &+ (r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi)^2 - ( \\ &r\sinh(\theta)\cos(\phi)\,(r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi) \\ &+ r\sinh(\theta)\sin(\phi)\,(r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi))^2 \Big/ ( \\ &r^2 + r^2\sinh(\theta)^2\cos(\phi)^2 + r^2\sinh(\theta)^2\sin(\phi)^2) \end{aligned}$$

```
> expand(ds2);
```

$$\begin{aligned} &r^2\cosh(\theta)^2\cos(\phi)^2\,dtheta^2 + r^2\sinh(\theta)^2\sin(\phi)^2\,dphi^2 + r^2\cosh(\theta)^2\sin(\phi)^2\,dtheta^2 \\ &+ r^2\sinh(\theta)^2\cos(\phi)^2\,dphi^2 - \frac{r^4\sinh(\theta)^2\cos(\phi)^4\cosh(\theta)^2\,dtheta^2}{\%1} \\ &- 2\,\frac{r^4\sinh(\theta)^2\cos(\phi)^2\cosh(\theta)^2\,dtheta^2\sin(\phi)^2}{\%1} - \frac{r^4\sinh(\theta)^2\sin(\phi)^4\cosh(\theta)^2\,dtheta^2}{\%1} \\ &\%1 := r^2 + r^2\sinh(\theta)^2\cos(\phi)^2 + r^2\sinh(\theta)^2\sin(\phi)^2 \end{aligned}$$

```
> simplify(",{cosh(theta)^2-sinh(theta)^2=1}, [cosh(theta)]);
```

$$r^2\,dtheta^2 + r^2\sinh(\theta)^2\,dphi^2$$

## 2D Metrics

To summarize:

$$\text{Sphere:}\quad ds^2 = R^2\left\{\frac{dr^2}{1-r^2} + r^2\,d\theta^2\right\} \tag{4.12}$$

$$\text{Plane:}\quad ds^2 = R^2\left\{dr^2 + r^2\,d\theta^2\right\} \tag{4.6}$$

$$\text{Hyperbolic Plane:}\quad ds^2 = R^2\left\{\frac{dr^2}{1+r^2} + r^2\,d\theta^2\right\} \tag{4.18}$$

$\Longrightarrow$ All three metrics can be written as

$$ds^2 = R^2\left\{\frac{dr^2}{1-k\,r^2} + r^2\,d\theta^2\right\} \tag{4.21}$$

where $k$ defines the geometry:

$$k = \begin{cases} +1 & \text{spherical} \\ 0 & \text{planar} \\ -1 & \text{hyperbolic} \end{cases} \tag{4.22}$$





## 2D Metrics

For "spherical coordinates" we found:

$$\text{Sphere:}\quad ds^2 = R^2\left\{d\theta^2 + \sin^2\theta\,d\phi^2\right\} \tag{4.14}$$

$$\text{Plane:}\quad ds^2 = R^2\left\{d\theta^2 + \theta^2 d\phi^2\right\} \tag{4.6}$$

$$\text{Hyperbolic Plane:}\quad ds^2 = R^2\left\{d\theta^2 + \sinh^2\theta\,d\phi^2\right\} \tag{4.20}$$

$\Longrightarrow$ All three metrics can be written as

$$ds^2 = R^2\left\{d\theta^2 + S_k^2(\theta)\,d\phi^2\right\} \tag{4.23}$$

where

$$S_k(\theta) = \begin{cases} \sin\theta & \text{for } k=+1 \\ \theta & \text{for } k=\phantom{+}0 \\ \sinh\theta & \text{for } k=-1 \end{cases} \quad\text{and}\quad C_k(\theta) = \sqrt{1-kS_k^2(\theta)} = \begin{cases} \cos\theta & \text{for } k=+1 \\ 1 & \text{for } k=\phantom{+}0 \\ \cosh\theta & \text{for } k=-1 \end{cases} \tag{4.24}$$

The cos-like analogue of $S_k$, $C_k$, will be needed later

Note that, compared to the earlier formulae, some coordinates have been renamed. This is confusing, but legal...





## RW Metric

- Cosmological principle $+$ expansion $\Longrightarrow\exists$ freely expanding cosmical coordinate system.
  - Observers $=:$ fundamental observers
  - Time $=:$ cosmic time

  This is the coordinate system in which the 3K radiation is isotropic, clocks can be synchronized, e.g., by adjusting time to the local density of the universe.

  $\Longrightarrow$ Metric has temporal and spatial part.

  This also follows directly from the equivalence principle.
- *Homogeneity and isotropy* $\Longrightarrow$ spatial part is spherically symmetric:

$$d\psi^2 := d\theta^2 + \sin^2\theta\,d\phi^2 \tag{4.25}$$

- *Expansion:* $\exists$ scale factor, $R(t)$ $\Longrightarrow$ measure distances using comoving coordinates.

$\Longrightarrow$ metric looks like

$$ds^2 = c^2\,dt^2 - R^2(t)\left[f^2(r)\,dr^2 + g^2(r)\,d\psi^2\right] \tag{4.26}$$

where $f(r)$ and $g(r)$ are arbitrary.





## RW Metric

Metrics of the form of eq. (4.26) are called Robertson-Walker (RW) metrics (introduced in 1935).

Previously studied by Friedmann and Lemaître...

One common choice is

$$\boxed{ds^2 = c^2\,dt^2 - R^2(t)\left[dr^2 + S_k^2(r)\,d\psi^2\right]} \tag{4.27}$$

where

$R(t)$: scale factor, containing the physics

$t$: cosmic time

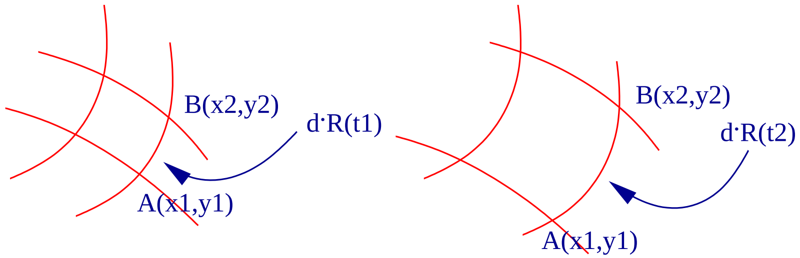
$r$, $\theta$, $\phi$: comoving coordinates

$S_k(r)$ was defined in Eq. (4.24).

Remark: $\theta$ and $\phi$ describe directions on sky, as seen from the arbitrary center of the coordinate system (=us), $r$ can be interpreted as a radial coordinate.

## RW Metric

The RW metric defines an universal coordinate system tied to expansion of space:




Scale factor $R(t)$ describes evolution of universe.

- $d$ is called the comoving distance.
- $D(t) := d \cdot R(t)$ is called the proper distance,

(note that $R$ is unitless, i.e., $d$ and $d \cdot R(t)$ are measured in Mpc)





## RW Metric

Other forms of the RW metric are also used:

1. Substitution $S_k(r) \longrightarrow r$ gives

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\psi^2\right\} \tag{4.28}$$

(i.e., other definition of comoving radius $r$).

2. A metric with a **dimensionless scale factor**,

$$a(t) := \frac{R(t)}{R(t_0)} = \frac{R(t)}{R_0} \tag{4.29}$$

(where $t_0$=today, i.e., $a(t_0) = 1$), gives

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left\{\mathrm{d}r^2 + \frac{S_k^2(R_0 r)}{R_0^2}\,\mathrm{d}\psi^2\right\} \tag{4.30}$$





## RW Metric

3. Using $a(t)$ *and* the substitution $S_k(r) \longrightarrow r$ is also possible:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left\{\frac{\mathrm{d}r^2}{1-k\cdot(R_0 r)^2} + r^2\,\mathrm{d}\psi^2\right\} \tag{4.31}$$

The units of $R_0 r$ are Mpc $\Longrightarrow$ *Used for observations!*

4. Replace cosmic time, $t$, by **conformal time**, $\mathrm{d}\eta = \mathrm{d}t/R(t)$ $\Longrightarrow$ conformal metric,

$$\mathrm{d}s^2 = R^2(\eta)\left\{\mathrm{d}\eta^2 - \frac{\mathrm{d}r^2}{1-kr} - r^2\,\mathrm{d}\psi^2\right\} \tag{4.32}$$

Theoretical importance of this metric: For $k = 0$, i.e., a flat space, the RW metric = Minkowski line element $\times\ R^2(\eta) \Longrightarrow$ Equivalence principle!





## RW Metric

5. Finally, the metric can also be written in the **isotropic form**,

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \frac{R(t)}{1+(k/4)r^2}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\psi^2\right\} \tag{4.33}$$

Here, the term in $\{\dots\}$ is just the line element of a 3d-sphere $\Longrightarrow$ isotropy!

*Note:* There are as many notations as authors, e.g., some use $a(t)$ where we use $R(t)$, etc. $\Longrightarrow$ Be careful!

*Note 2: Local* homogeneity and isotropy (i.e., within a Hubble radius, $r = c/H_0$), do not imply *global* homogeneity and isotropy $\Longrightarrow$ Cosmologies with a **non-trivial topology** are possible (e.g., also with more dimensions...).