Hubble's Law

Hubble's Law follows from the variation of $R(t)$:

Small scales $\Longrightarrow$ Euclidean geometry. Then the proper distance between two observers is:

$$D(t) = d \cdot R(t) \tag{4.34}$$

where d: comoving distance.

Expansion $\Longrightarrow$ proper separation changes:

$$\frac{\Delta D}{\Delta t} = \frac{R(t+\Delta t)d - R(t)d}{\Delta t} \Longrightarrow \lim_{\Delta t \to 0} \Longrightarrow \quad v = \frac{\mathrm{d}D}{\mathrm{d}t} = \dot{R}\, d = \frac{\dot{R}}{R} D =: H\, D \tag{4.35}$$

$\Longrightarrow$ Identify local Hubble "constant" as

$$H = \frac{\dot{R}}{R} = \dot{a}(t) \quad (a(t) \text{ from Eq. 4.29, } a(\text{today}) = 1) \tag{4.36}$$

Since $R = R(t) \Longrightarrow H$ is time-dependent!

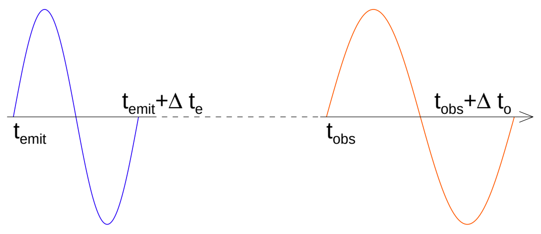
Note that the definition of H allows us to derive Hubble's relation for the case of small v, i.e., $v \ll c$. In this case, the red-shift is

$$z = \frac{v}{c} \quad \Longrightarrow \quad z = \frac{Hd}{c} \tag{4.41}$$

An alternative derivation of the cosmological redshift follows directly from general relativity, using the basic GR fact that for photons $\mathrm{d}s^2 = 0$. Inserting this into the metric, and assuming without loss of generality that $\mathrm{d}\psi^2 = 0$, one finds

$$0 = c^2\,\mathrm{d}t^2 - R^2(t)\,\mathrm{d}r^2 \quad \Longrightarrow \quad \mathrm{d}r = \pm \frac{c\,\mathrm{d}t}{R(t)} \tag{4.42}$$

Since photons travel forward, we choose the $+$-sign.



The *comoving* distance traveled by photons emitted at cosmic times t_{emit} and $t_{\mathrm{emit}} + \Delta t_{\mathrm{e}}$ is

$$r_1 = \int_{t_{\mathrm{emit}}}^{t_{\mathrm{obs}}} \frac{c\,\mathrm{d}t}{R(t)} \quad \text{and} \quad r_2 = \int_{t_{\mathrm{emit}}+\Delta t_{\mathrm{e}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{o}}} \frac{c\,\mathrm{d}t}{R(t)} \tag{4.43}$$

Redshift, I

The cosmological redshift is a consequence of the expansion of the universe:

The comoving distance is constant, thus in terms of the proper distance:

$$d = \frac{D(t = \text{today})}{R(t = \text{today})} = \frac{D(t)}{R(t)} = \text{const.} \tag{4.37}$$

Set $a(t) = R(t)/R(t = \text{today})$, then eq. (4.37) implies

$$\lambda_{\text{obs}} = \frac{\lambda_{\text{emit}}}{a_{\text{emit}}} \tag{4.38}$$

(λ_{obs}: observed wavelength, λ_{emit}: emitted wavelength)

Thus the observed redshift is

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} - 1 = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} - 1 \tag{4.39}$$

$$\Longrightarrow \quad 1 + z = \frac{1}{a_{\text{emit}}} = \frac{R(t = \text{today})}{R(t)} = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} \tag{4.40}$$

Light emitted at $z = 1$ was emitted when the universe was half as big as today!

z: measure for *relative size* of universe at time the observed light was emitted.

But the comoving distances are equal, $r_1 = r_2$! Therefore

$$0 = \int_{t_{\mathrm{emit}}}^{t_{\mathrm{obs}}} \frac{c\,\mathrm{d}t}{R(t)} - \int_{t_{\mathrm{emit}}+\Delta t_{\mathrm{e}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{o}}} \frac{c\,\mathrm{d}t}{R(t)} \tag{4.44}$$

$$= \int_{t_{\mathrm{emit}}}^{t_{\mathrm{emit}}+\Delta t_{\mathrm{e}}} \frac{c\,\mathrm{d}t}{R(t)} - \int_{t_{\mathrm{obs}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{o}}} \frac{c\,\mathrm{d}t}{R(t)} \tag{4.45}$$

If Δt small $\Longrightarrow R(t) \approx$ const.:

$$= \frac{c\,\Delta t_{\mathrm{e}}}{R(t_{\mathrm{emit}})} - \frac{c\,\Delta t_{\mathrm{o}}}{R(t_{\mathrm{obs}})} \tag{4.46}$$

For a wave: $c\Delta t = \lambda$, such that

$$\frac{\lambda_{\mathrm{emit}}}{R(t_{\mathrm{emit}})} = \frac{\lambda_{\mathrm{obs}}}{R(t_{\mathrm{obs}})} \quad \Longleftrightarrow \quad \frac{\lambda_{\mathrm{emit}}}{\lambda_{\mathrm{obs}}} = \frac{R(t_{\mathrm{emit}})}{R(t_{\mathrm{obs}})} \tag{4.47}$$

From this equation it is straightforward to derive Eq. (4.39).

Redshift, II

Outside of the local universe: Eq. (4.40) only valid interpretation of z.

$\Longrightarrow$ It is common to interpret z as in special relativity:

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{4.48}$$

This is WRONG

Redshift is due to expansion of space, not due to motion of galaxy.

What *is* true is that z is accumulation of many infinitesimal red-shifts à la Eq. (4.41), see, e.g., Peacock (1999).

Time Dilatation

For light, $D = c\,\Delta t$. Then a consequence of Eq. (4.37) is

$$\frac{c\,\Delta t_{\text{emit}}}{R(t_{\text{emit}})} = \frac{c\,\Delta t_{\text{obs}}}{R(t_{\text{obs}})} \quad \Longrightarrow \quad \frac{\mathrm{d}t}{R} = \text{const.} \tag{4.46}$$

In other words:

$$\frac{\mathrm{d}t_{\text{obs}}}{\mathrm{d}t_{\text{emit}}} = \frac{R(t_{\text{obs}})}{R(t_{\text{emit}})} = 1 + z \tag{4.49}$$

$\Longrightarrow$ Time dilatation of events at large z.

This cosmological time dilatation has been observed in the light curves of supernova outbursts.

All other observables apart from z (e.g., number density $N(z)$, luminosity distance d_{L}, etc.) require explicit knowledge of $R(t)$

$\Longrightarrow$ Need to look at the dynamics of the universe.

Friedmann Equations, I

General relativistic approach: Insert metric into Einstein equation to obtain differential equation for $R(t)$:

Einstein equation:

$$\underbrace{R_{\mu\nu} - \frac{1}{2}\mathscr{R} g_{\mu\nu}}_{G_{\mu\nu}} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{4.50}$$

where

$g_{\mu\nu}$: Metric tensor ($\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$)

$R_{\mu\nu}$: Ricci tensor (function of $g_{\mu\nu}$)

$\mathscr{R}$: Ricci scalar (function of $g_{\mu\nu}$)

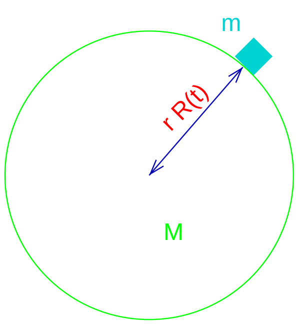
$G_{\mu\nu}$: Einstein tensor (function of $g_{\mu\nu}$)

$T_{\mu\nu}$: Stress-energy tensor, describing curvature of space due to fields present (matter, radiation,...)

Λ: Cosmological constant

$\Longrightarrow$ Messy, but doable

Friedmann Equations, II



Here, Newtonian derivation of Friedmann equations: Dynamics of a mass element on the surface of sphere of density $\rho(t)$ and comoving radius d, i.e., proper radius $d \cdot R(t)$ (McCrea, 1937)

Mass of sphere:

$$M = \frac{4\pi}{3}(d\,R)^3\rho(t) = \frac{4\pi}{3}d^3\rho_0 \quad \text{where} \quad \rho(t) = \frac{\rho_0}{R(t)^3} \tag{4.51}$$

Force on mass element:

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(d\,R(t)) = -\frac{GMm}{(dR(t))^2} = -\frac{4\pi G}{3}\frac{d\rho_0}{R^2(t)}m \tag{4.52}$$

Canceling $m \cdot d$ gives momentum equation:

$$\ddot{R}(t) = -\frac{4\pi G}{3}\frac{\rho_0}{R(t)^2} = -\frac{4\pi G}{3}\rho(t)R(t) \tag{4.53}$$

Multiplying Eq. (4.53) with $\dot{R}$ and integrating yields the energy equation:

$$\frac{1}{2}\dot{R}(t)^2 = +\frac{4\pi G}{3}\frac{\rho_0}{R(t)} + \text{const.} = +\frac{4\pi G}{3}\rho(t)R^2(t) + \text{const.} \tag{4.54}$$

where the constant can only be obtained from GR.

Friedmann Equations, III

Problems with the Newtonian derivation:

1. Cloud is implicitly assumed to have $r_{\text{cloud}} < \infty$

   (for $r_{\text{cloud}} \to \infty$ the force is undefined)

   $\Longrightarrow$ violates cosmological principle.

2. Particles move *through* space

   $\Longrightarrow v > c$ possible

   $\Longrightarrow$ violates SRT.

Why do we get correct result?

GRT $\longrightarrow$ Newton for small scales and mass densities

Since universe is isotropic: scale invariance on Mpc scales

$\Longrightarrow$ Newton sufficient (classical limit of GR).

(In fact, point 1 above *does* hold in GR: Birkhoff's theorem).

Friedmann Equations, IV

The exact GR derivation of Friedmanns equation gives:

$$\ddot{R} = -\frac{4\pi G}{3} R \left(\rho + \frac{3p}{c^2}\right) + \left[\frac{1}{3}\Lambda R\right]$$
$$\dot{R}^2 = +\frac{8\pi G \rho}{3} R^2 - kc^2 + \left[\frac{1}{3}\Lambda c^2 R^2\right] \qquad (4.55)$$

Notes:

1. For $k = 0$: Eq. (4.55) $\longrightarrow$ Eq. (4.54).
2. k determines the curvature of space (and is *not* an integer here!).
3. The density, ρ, includes the contribution of all different kinds of energy (remember mass-energy equivalence!).
4. There is energy associated with the vacuum, parameterized by the parameter Λ.

The evolution of the Hubble parameter is ($\Lambda = 0$):

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2(t) = \frac{8\pi G \rho}{3} - \frac{kc^2}{R^2} \qquad (4.56)$$

The Critical Density, I

Solving Eq. (4.56) for k:

$$\frac{R^2}{c}\left(\frac{8\pi G}{3}\rho - H^2\right) = k \qquad (4.57)$$

$\Longrightarrow$ Sign of curvature parameter k only depends on density, ρ. With

$$\rho_c = \frac{3H^2}{8\pi G} \quad \text{and} \quad \Omega = \frac{\rho}{\rho_c} \qquad (4.58)$$

it is easy to see that:

$\Omega > 1 \Longrightarrow k > 0 \Longrightarrow$ closed universe

$\Omega = 1 \Longrightarrow k = 0 \Longrightarrow$ flat universe

$\Omega < 1 \Longrightarrow k < 0 \Longrightarrow$ open universe

ρ_c is called the critical density

For $\Omega \leq 1$ the universe will expand until ∞,

For $\Omega > 1$ we will see the "big crunch".

Current value of ρ_c: $\sim 1.67 \times 10^{-24}\,\text{g}\,\text{cm}^{-3}$ (3...10 H-atoms m^{-3}).

The Critical Density, II

Ω has a second order effect on the expansion:

Taylor series of $R(t)$ around $t = t_0$:

$$\frac{R(t)}{R(t_0)} = \frac{R(t_0)}{R(t_0)} + \frac{\dot{R}(t_0)}{R(t_0)}(t - t_0) + \frac{1}{2}\frac{\ddot{R}(t_0)}{R(t_0)}(t - t_0)^2 \qquad (4.59)$$

The Friedmann equation Eq. (4.53) can be written

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho = -\frac{4\pi G}{3}\Omega\frac{3H^2}{8\pi G} = -\frac{\Omega H^2}{2} \qquad (4.60)$$

Since $H(t) = \dot{R}/R$ (Eq. 4.36), Eq. (4.59) is

$$\frac{R(t)}{R(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}\frac{\Omega_0}{2}H_0^2(t - t_0)^2 \qquad (4.61)$$

where $H_0 = H(t_0)$ and $\Omega_0 = \Omega(t_0)$.

The subscript 0 is often omitted in the case of Ω.

Often, Eq. (4.61) is written using the deceleration parameter:

$$q := \frac{\Omega}{2} = -\frac{\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)} \qquad (4.62)$$

Equation of state, I

Evolution of the universe determined by **three different kinds of equation of state**:

1. **Matter**: Normal (nonrelativistic) particles get **diluted by expansion** of the universe:

$$\rho_{\mathrm{m}} \propto R^{-3} \tag{4.63}$$

Matter is also often called **dust** by cosmologists.

2. **Radiation**: The energy density of radiation decreases because of **volume expansion** and because of the **cosmological redshift** (Eq. 4.47: $\lambda_{\mathrm{obs}}/\lambda_{\mathrm{emit}} = \nu_{\mathrm{emit}}/\nu_{\mathrm{obs}} = R(t_{\mathrm{obs}})/R(t_{\mathrm{emit}})$) such that

$$\rho_{\mathrm{r}} \propto R^{-4} \tag{4.64}$$

3. **Vacuum**: The vacuum energy density (=Λ) is **independent of R**:

$$\rho_{\mathrm{v}} = \text{const.} \tag{4.65}$$

Inserting these equations of state into the Friedmann equation and solving with the boundary condition $R(t=0)=0$ then gives a specific world model.

Equation of state, II

Current scale factor is determined by H_0 and Ω_0:

Friedmann for $t = t_0$:

$$\dot{R}_0^2 - \frac{8\pi G}{3}\rho R_0^2 = -kc^2 \tag{4.66}$$

Insert Ω and note $H_0 = \dot{R}_0/R_0$

$$\Longleftrightarrow \quad H_0^2 R_0^2 - H_0^2 \Omega_0 R_0^2 = -kc^2 \tag{4.67}$$

And therefore

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega - 1}} \tag{4.68}$$

For $\Omega \longrightarrow 0$, $R_0 \longrightarrow c/H_0$, the **Hubble length**.
For $\Omega = 1$, R_0 is arbitrary.

We now have everything we need to solve the Friedmann equation and determine the evolution of the universe for $k = 0, +1$, and -1.

$k = 0$, Matter dominated

For the **matter dominated**, **flat** case (the **Einstein-de Sitter case**), the Friedmann equation is

$$\dot{R}^2 - \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R^3} R^2 = 0 \tag{4.69}$$

For $k = 0$: $\Omega = 1$ and

$$\frac{8\pi G \rho_0}{3} = \Omega_0 H_0^2 R_0^3 = H_0^2 R_0^3 \tag{4.70}$$

Therefore, the Friedmann eq. is

$$\dot{R}^2 - \frac{H_0^2 R_0^3}{R} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}R}{\mathrm{d}t} = H_0 R_0^{3/2} R^{-1/2} \tag{4.71}$$

Separation of variables and setting $R(0) = 0$,

$$\int_0^{R(t)} R^{1/2}\,\mathrm{d}R = H_0 R_0^{3/2} t \quad \Longrightarrow \quad \frac{2}{3}R^{3/2}(t) = H_0 R_0^{3/2} t \quad \Longrightarrow \quad R(t) = R_0 \left(\frac{3H_0}{2}\,t\right)^{2/3} \tag{4.72}$$

Therefore, for $k = 0$, the universe expands until ∞, its **current age** ($R(t_0) = R_0$) is given by

$$t_0 = \frac{2}{3H_0} \tag{4.73}$$

Reminder: The Hubble-Time is $H_0^{-1} = 9.78\,\mathrm{Gyr}/h$.

For the **matter dominated**, **closed** case, Friedmanns equation is

$$\dot{R}^2 - \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R} = -c^2 \quad \Longleftrightarrow \quad \dot{R}^2 - \frac{H_0^2 R_0^3 \Omega_0}{R} = -c^2 \tag{4.74}$$

Inserting R_0 from Eq. (4.68) gives

$$\dot{R}^2 - \frac{H_0^2 c^3 \Omega_0}{H_0^3(\Omega - 1)^{3/2}}\frac{1}{R} = -c^2 \tag{4.75}$$

which is equivalent to

$$\frac{\mathrm{d}R}{\mathrm{d}t} = c\left(\frac{\xi}{R} - 1\right)^{1/2} \quad \text{with} \quad \xi = \frac{c}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{4.76}$$

With the boundary condition $R(0) = 0$, separation of variables gives

$$ct = \int_0^{R(t)} \frac{\mathrm{d}R}{(\xi/R - 1)^{1/2}} = \int_0^{R(t)} \frac{\sqrt{R}\,\mathrm{d}R}{(\xi - R)^{1/2}} \tag{4.77}$$

Integration by substitution gives the "cycloid solution"

$$R = \xi \sin^2\frac{\theta}{2} = \frac{\xi}{2}(1 - \cos\theta) \quad \text{and} \quad ct = \frac{\xi}{2}(\theta - \sin\theta) \tag{4.78}$$

where θ is an implicit parameter.

The age of the universe, t_0, is obtained by solving

$$R_0 = \frac{c}{H_0(\Omega_0 - 1)^{1/2}} = \frac{\xi}{2}(1 - \cos\theta_0) = \frac{1}{2}\frac{c}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(1 - \cos\theta_0) \tag{4.79}$$

(remember Eq. 4.68!). Therefore

$$\cos\theta_0 = \frac{2 - \Omega_0}{\Omega_0} \quad \Longleftrightarrow \quad \sin\theta_0 = \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1} \tag{4.80}$$

Inserting this into Eq. (4.78) gives

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left[\arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1}\right] \tag{4.81}$$

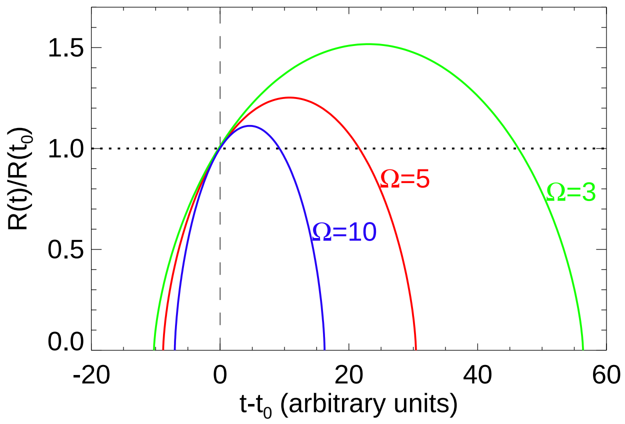
The cycloid solution shows that for $\Omega > 1$, the universe has a finite lifetime, i.e., it expands to a maximum and then becomes smaller and dies in a "big crunch". The max. expansion occurs at $\theta = \pi$, with a maximum scale factor of

$$R_{\text{max}} = \xi = \frac{c}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{4.82}$$

The big crunch will happen at $\theta = 2\pi$, such that the lifetime of the closed universe is

$$t_{\text{life}} = \frac{\pi}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{4.83}$$

$k = +1$, Matter dominated, I



For the closed universe, one finds

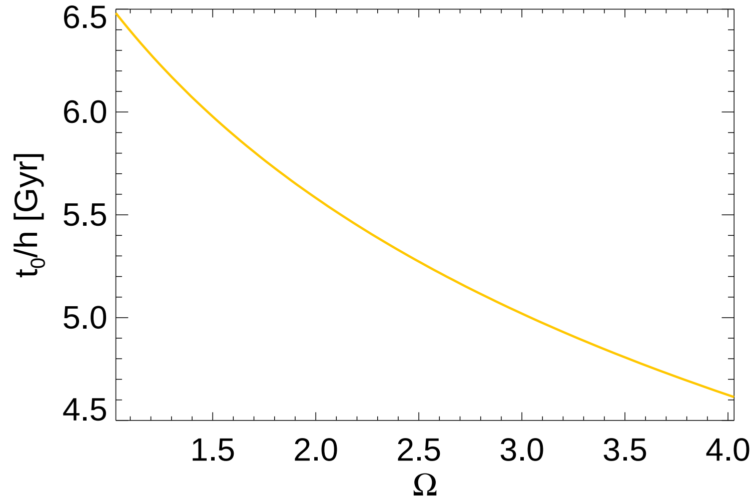
$$R = \frac{\xi}{2}(1 - \cos\theta)$$
$$ct = \frac{\xi}{2}(\theta - \sin\theta) \tag{4.78}$$

Note that R is a cyclic function

$\Longrightarrow$ The closed universe has a finite lifetime, given by

$$t_{\text{life}} = \frac{\pi}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{4.83}$$

$k = +1$, Matter dominated, II



Age of a closed and matter dominated universe.

$k = -1$, Matter dominated, I

Finally, the **matter dominated, open** case. This case is very similar to the case of $k = +1$:

For $k = -1$, the Friedmann equation becomes

$$\frac{\mathrm{d}R}{\mathrm{d}t} = c\left(\frac{\zeta}{R} + 1\right)^{1/2} \tag{4.84}$$

where

$$\zeta = \frac{c}{H_0} \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} \tag{4.85}$$

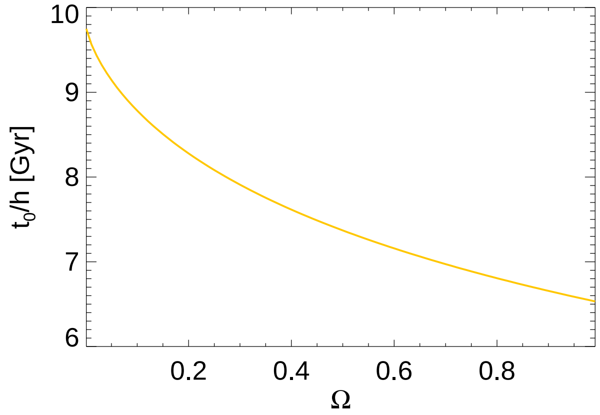
Separation of variables gives after a little bit of algebra

$$R = \frac{\zeta}{2}(\cosh\theta - 1)$$
$$ct = \frac{\zeta}{2}(\sinh\theta - 1) \tag{4.86}$$

where the integration was again performed by substitution.

Note: θ here has *nothing* to do with the coordinate angle θ!

$k = -1$, Matter dominated, II



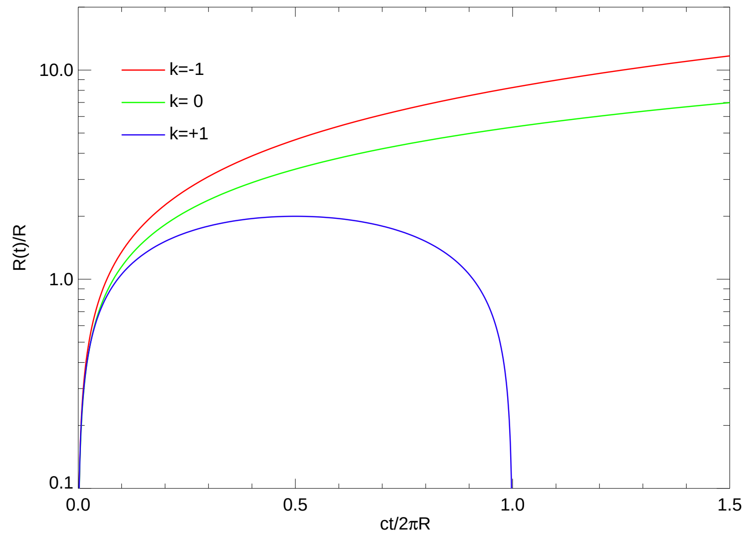
To obtain the age of the universe, note that at the present time,

$$\cosh\theta_0 = \frac{2-\Omega_0}{\Omega_0}$$
$$\sinh\theta_0 = \frac{2}{\Omega_0}\sqrt{1-\Omega_0} \tag{4.87}$$

(identical derivation as that leading to Eq. 4.79)

therefore,

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\cdot\left\{\frac{2}{\Omega_0}\sqrt{1-\Omega_0} - \ln\left(\frac{2-\Omega_0+2\sqrt{1-\Omega_0}}{\Omega_0}\right)\right\} \tag{4.88}$$



Summary

For the matter dominated case, our results from Eqs. (4.78), and (4.86) can be written with the functions S_k and C_k (Eq. 4.24) in form of the **cycloid solution**:

$$R = k\mathscr{R}\,(1 - C_k(\theta)) \qquad ct = k\mathscr{R}\,(\theta - S_k(\theta)) \tag{4.89}$$

with

$$S_k(\theta) = \begin{cases}\sin\theta \\ \theta \\ \sinh\theta\end{cases} \quad\text{and}\quad C_k(\theta) = \begin{cases}\cos\theta & \text{for } k=+1 \\ 1 & \text{for } k=0 \\ \cosh\theta & \text{for } k=-1\end{cases} \tag{4.24}$$

and where the **characteristic radius**, $\mathscr{R}$, is given by

$$\mathscr{R} = \frac{c}{H_0}\frac{\Omega_0/2}{(k(\Omega_0-1))^{3/2}} \tag{4.90}$$

Notes:

1. Eq. (4.89) can also be derived as the result of the Newtonian collapse/expansion of a spherical mass distribution.
2. θ is called the **development angle**, it is equal to the *conformal time* (Eq. (4.32)).

4–39

McCrea, W. H., & Milne, E. A., 1934, Quart. J. Math. (Oxford Series), 5, 73

Silk, J., 1997, A Short History of the Universe, Scientific American Library 53, (New York: W. H. Freeman)

Classical Cosmology

Classical Cosmology

To understand what universe we live in, we need to determine observationally the following numbers:

1. The Hubble constant, H_0
   $\Longrightarrow$ Requires distance measurements.
2. The current density parameter, Ω_0
   $\Longrightarrow$ Requires measurement of the mass density.
3. The cosmological constant, Λ
   $\Longrightarrow$ Requires acceleration measurements.
4. The age of the universe, t_0, for consistency checks
   $\Longrightarrow$ Requires age measurements.

The determination of these numbers is the realm of classical cosmology.

First part: Distance determination and H_0!

Introduction, I

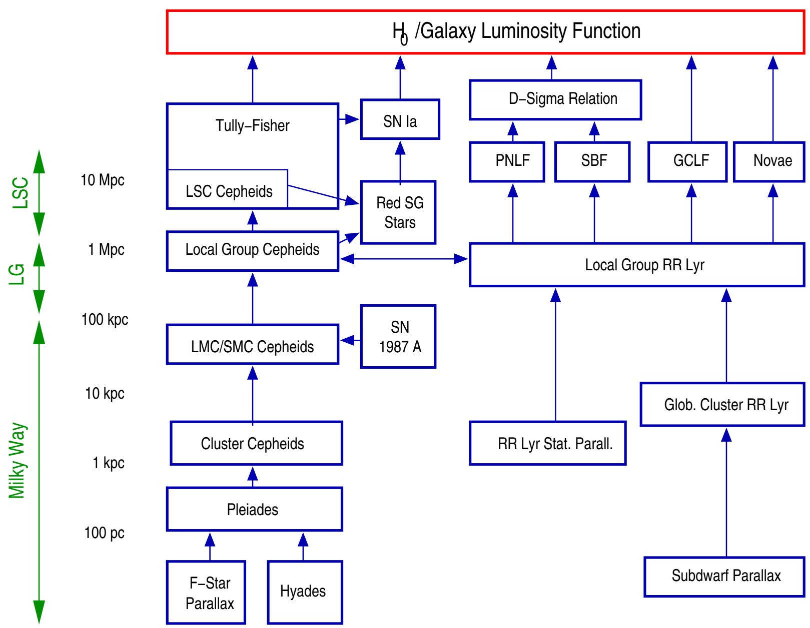
Distances are required for determination of H_0.

$\Longrightarrow$ Need to measure distances out to $\sim$200 Mpc to obtain reliable values.

To get this far: cosmological distance ladder.

1. Trigonometric Parallax and Moving Cluster
2. Main Sequence Fitting
3. RR Lyr
4. Baade-Wesselink
5. Cepheids
6. (Light echos)
7. Brightest Stars
8. Type Ia Supernovae
9. Tully-Fisher
10. D_n-σ for ellipticals
11. Brightest Cluster Galaxies
12. Gravitational Lenses

Still the best reference on this subject is ROWAN-ROBINSON, M., 1985, The Cosmological Distance Ladder, New York: Freeman.



(after Jacoby et al., 1992, Fig. 1)

Units

Basic unit of length in astronomy: Astronomical Unit (AU).

Colloquial Definition: 1 AU = mean distance Earth–Sun.

Measurement: (Venus) radar ranging, interplanetary satellite positions, χ^2 minimization of N-body simulations of solar system

1 AU ∼ 149.6 × 10⁶ km → $1\,\mathrm{AU} \sim 149.6 \times 10^6\,\mathrm{km}$

In the astronomical system of units (IAU 1976), the AU is defined via Gaussian gravitational constant (k), where the acceleration

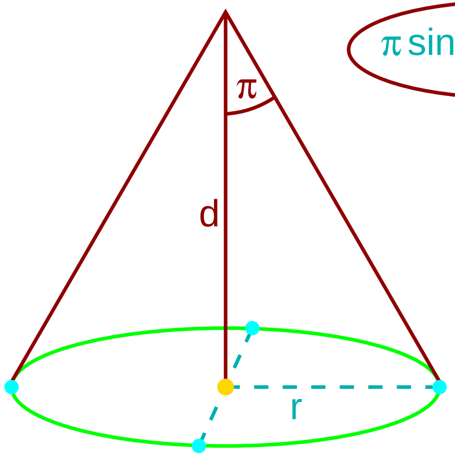
$$\ddot{\boldsymbol{r}} = -\frac{k^2(1+m)\boldsymbol{r}}{r^3}$$

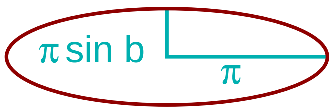
where $k := 0.01720209895$, leading to $a_\oplus = 1.00000105726665$, and 1 AU=$1.4959787066 \times 10^{11}$ m (Seidelmann, 1992).

Reason for this definition: k much better known than G.

(2006 CODATA: $G = 6.67428(67) \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, so only known to 4 significant digits)

Trigonometric Parallax, I





Motion of Earth around Sun ⟹ Parallax produces apparent motion by amount

$$\tan \pi \sim \pi = r_\oplus / d \qquad (5.1)$$

π is called the trigonometric parallax, and *not* 3.141!

If star is at ecliptic latitude b, then ellipse with axes π and $\pi \sin b$.

Measurement difficult: $\pi \lesssim 0.76''$ (α Cen). Define unit for distance:

Parsec: Distance where 1 AU has $\pi = 1''$. 1 pc = 206265 AU = 3.08×10^{18} cm = 3.26 ly

after Rowan-Robinson (1985, Fig. 2.1)

Trigonometric Parallax, II

Best measurements to date: Hipparcos satellite (1989–1993)

- systematic error of position: ∼0.5 mas for stars brighter 9 mag
- effective distance limit: 1 kpc
- standard error of proper motion: ∼1 mas yr^{-1}
- broad band photometry
- narrow band: B − V, V − J
- magnitude limit: 12 mag
- complete to mag: 7.3–9.0

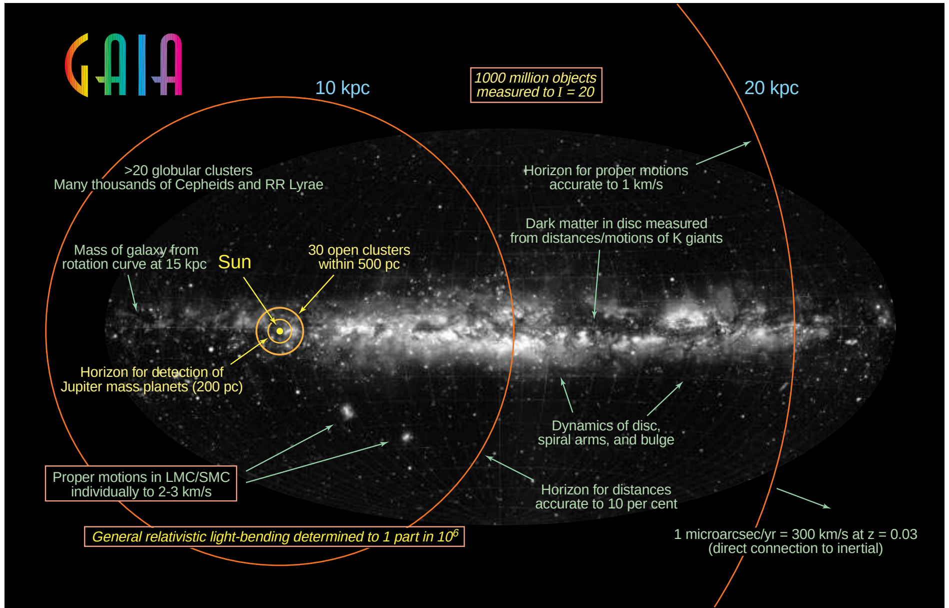
Results available at
`http://www.rssd.esa.int/index.php?project=HIPPARCOS`

Hipparcos catalogue: 118 218 objects with milliarcsecond precision.

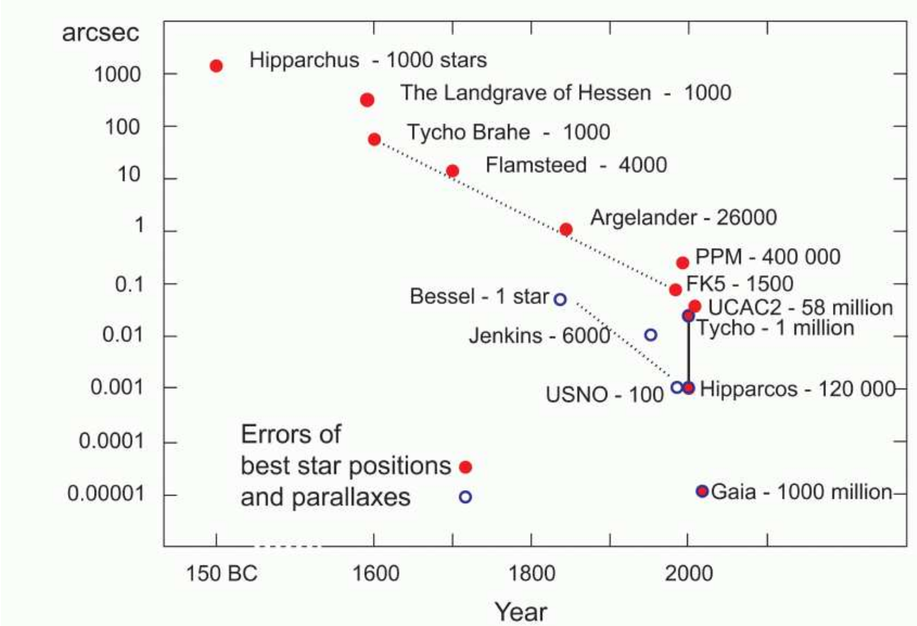
Tycho catalogue: 2 539 913 stars with 20–30 mas precision, two-band photometry (99% complete down to 11 mag)

Revised Hipparcos calibration: see van Leeuwen (2007).

GAIA (ESA mission, to be launched 2011 Dec on Soyuz from Kourou):

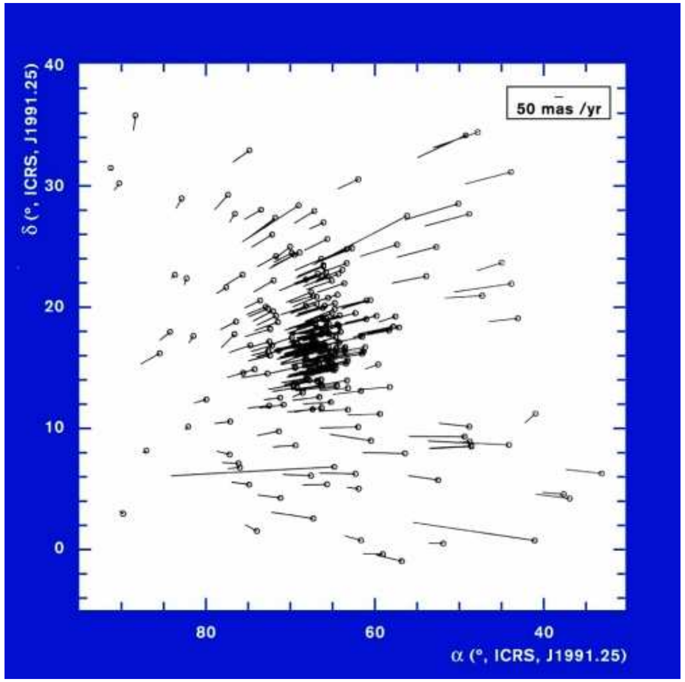


GAIA: ∼ 4μarcsec precision, 4 color to $V = 20$ mag, 10^9 objects.



ESA/M. Perryman

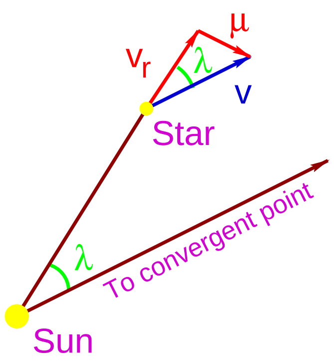
Development of the precision of astronomical position measurements



Source: ESA

© Till Credner

Moving Cluster



Perspective effect of spatial motion towards convergent point:

$$\tan\lambda = \frac{v_\mathrm{t}}{v_\mathrm{r}} = \frac{\mu d}{v_\mathrm{r}} \tag{5.2}$$

or

$$\frac{d}{1\,\mathrm{pc}} = \frac{v_\mathrm{r}/(1\,\mathrm{km/s})\tan\lambda}{4.74\,\mu/(1''/\mathrm{a})} \tag{5.3}$$

Problem: determination of convergent point

Less error prone: moving cluster method = rate of variation of angular diameter of cluster:

$$\dot{\theta} d = \theta v_\mathrm{r} \tag{5.4}$$

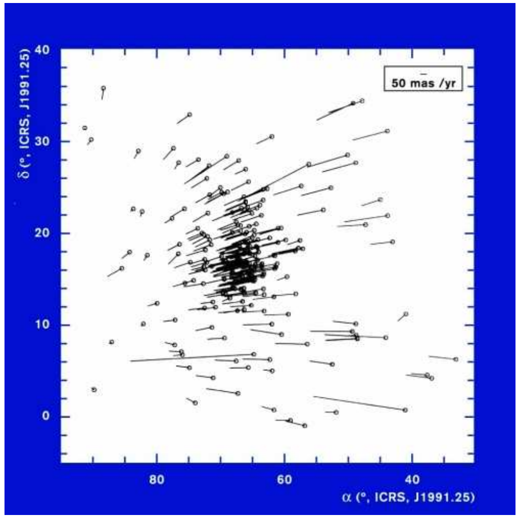
Observation of proper motions gives

$$\frac{\dot{\theta}}{\theta} = \frac{\mathrm{d}\mu_\alpha}{\mathrm{d}\alpha} = \frac{\mathrm{d}\mu_\delta}{\mathrm{d}\delta} \tag{5.5}$$

where $\mu_{\alpha,\delta}$ proper motion in α and δ. Therefore, from Eq. (5.4),

$$d = v_\mathrm{r}\,\frac{\theta}{\dot{\theta}} \tag{5.6}$$

Moving Cluster



Application: Distance to Hyades.
Tip of "arrow": Position of stars in 100000 years.
Hanson (1980) finds from this a distance of 46 pc

However: *Hipparcos:* geometric distance to Hyades is
$d = 46.34 \pm 0.27$ pc from parallax measurements.
$\Longrightarrow$ Moving cluster method only of historic interest.

Source: ESA

Interlude

Parallax and Moving Cluster: geometrical methods.

All other methods (exception: light echoes): standard candles.

Requirements for standard candles (Mould, Kennicutt, Jr. & Freedman, 2000):

1. Physical basis should be understood.
2. Parameters should be measurable objectively.
3. No corrections ("fudges") required.
4. Small intrinsic scatter ($\Longrightarrow$ requiring small number of measurements!).
5. Wide dynamic range in distance.

Magnitudes

Assuming isotropic emission, distance and luminosity are related ("inverse square law") $\Longrightarrow$ luminosity distance:

$$F = \frac{L}{4\pi d_\mathrm{L}^2} \tag{5.7}$$

where F is the measured flux (erg cm^{-2} s^{-1}) and L the luminosity (erg s^{-1}).
Definition also true for flux densities, I_ν (erg cm^{-2} s^{-1} Å^{-1}).

The magnitude is defined by

$$m = A - 2.5 \log_{10} F \tag{5.8}$$

where A is a constant used to define the zero point (defined by $m = 0$ mag for Vega).

For a filter with transmission function ϕ_ν,

$$m_i = A_i - 2.5 \log \int \phi_\nu F_\nu \, d\nu \tag{5.9}$$

where, e.g., $i =$ U, B, V.

Magnitudes

To enable comparison of luminosities: define

absolute magnitude M = magnitude at distance 10 pc

Thus, since $m = A - 2.5 \log(L/4\pi d^2)$,

$$M = m - 5 \log \left(\frac{d_\mathrm{L}}{10\,\mathrm{pc}} \right) \tag{5.10}$$

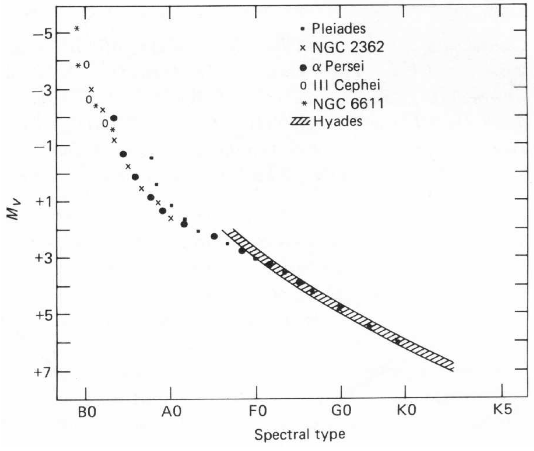
The difference $m - M$ is called the distance modulus, μ_0:

$$\mu_0 = \mathrm{DM} = m - M = 5 \log \left(\frac{d_\mathrm{L}}{10\,\mathrm{pc}} \right) \tag{5.11}$$

Often, distances are given in terms of $m - M$, and not in pc.

DM [mag]	3	5	10	15	20	25	30
d	40 pc	100 pc	1 kpc	10 kpc	100 kpc	1 Mpc	10 Mpc

Main Sequence Fitting, I



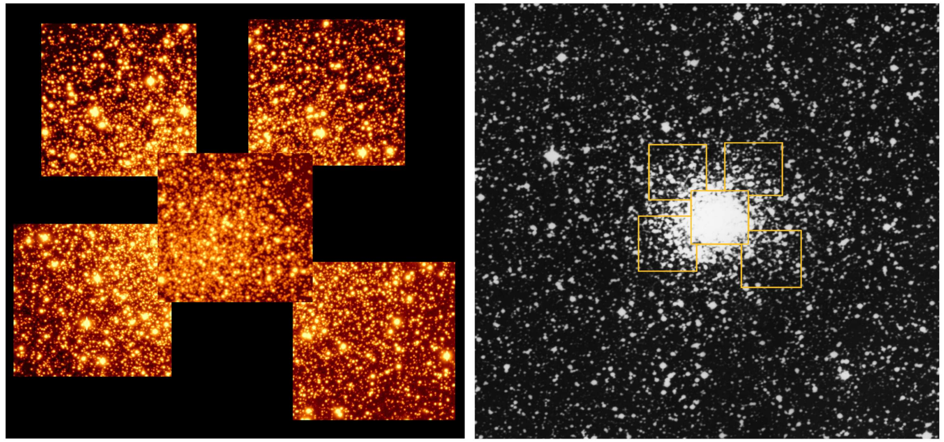
after Rowan-Robinson (1985, Fig. 2.11)

All open clusters are comparably young

$\Longrightarrow$ Hertzsprung Russell Diagram (HRD) dominated by Zero Age Main Sequence (ZAMS).

$\Longrightarrow$ Measure HRD (or Color Magnitude Diagram; CMD), shift magnitude scale until main sequence aligns

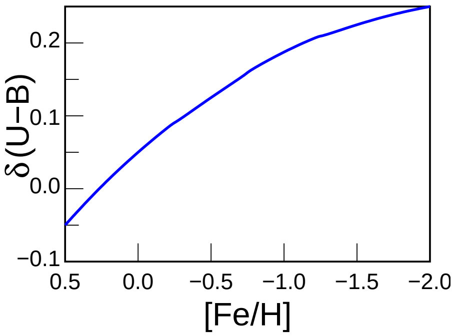
$\Longrightarrow$ distance modulus.



Globular Cluster NGC 6712

ESO PR Photo 06a/99 (18 February 1999)

Main Sequence Fitting, II



(after Rowan-Robinson, 1985, Fig. 2.12)

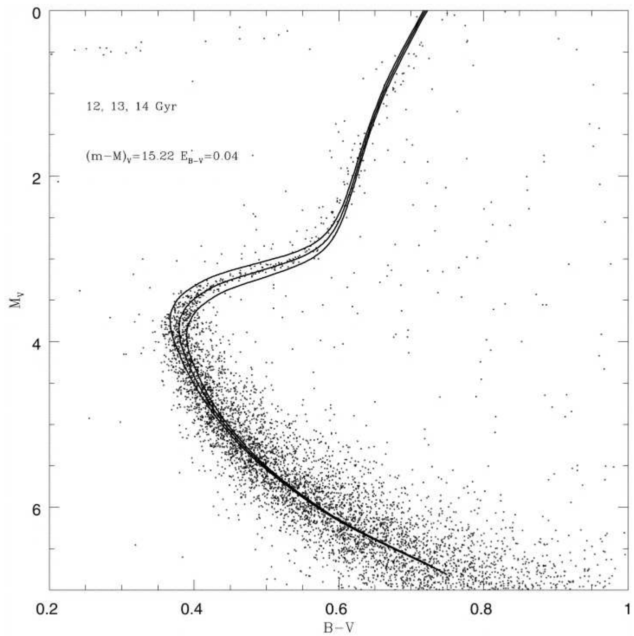
Caveats:

1. Location of ZAMS more age dependent than expected (van Leeuwen, 1999).
2. interstellar extinction $\Longrightarrow \mu_0 = \mu_V - A_V$, where μ_V, A_V DM/extinction measured in V-band.
3. metals: line blanketing (change in stellar continuum due to metal absorption lines, see figure) $\Longrightarrow$ Changes color $\Longrightarrow$ horizontal shift in CMD.

van den Bergh (1977): $Z_{\text{Hyades}} \sim 1.6 Z_\odot$, while other open clusters have solar metallicity $\Longrightarrow$ Cepheid DM were overestimated by 0.15 mag.

4. identification of unevolved stars crucial (evolution to larger magnitudes on MS during stellar life).

Currently: distances to ~200 open clusters known (Fenkart & Binggeli, 1979), limit ~7 kpc.



(M68, Straniero, Chieffi & Limongi, 1997, Fig. 11)

Globular clusters: HRD different from open clusters:

- population II $\Longrightarrow Z \ll Z_\odot$
- evolved

Use theoretical HRDs (isochrones) to obtain distance.

For distant clusters: MS unobservable $\Longrightarrow$ position of horizontal branch.

Baade-Wesselink

Basic principle (Baade, 1926): Assume black body

$\Longrightarrow$ Use color/spectrum to get kT_{eff}

$\Longrightarrow$ Emitted intensity is Planckian, B_ν

$\Longrightarrow$ Observed Intensity is $I_\nu \propto \pi R_*^2 \cdot B_\nu$.

Radius from integrating velocity profile of spectral lines:

$$R_2 - R_1 = p \int_1^2 v \, dt \tag{5.12}$$

(p: projection factor between velocity vector and line of sight).

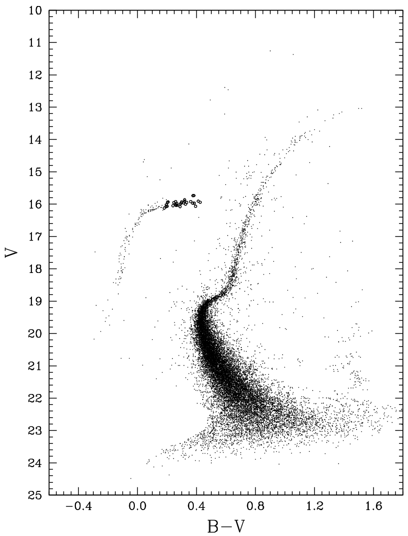
Wesselink (1947): Determine brightness for times of same color

$\Longrightarrow$ rather independent of knowledge of stellar spectrum (deviations from B_ν).

Stars: Calibration using interferometric diameters of nearby giants.

Baade-Wesselink works for pulsating stars such as RR Lyr, Cepheids, Miras, and expanding supernova remnants.

RR Lyr



RR Lyrae variables: Stars crossing instability strip in HRD

$\Longrightarrow$ Variability ($P \sim 0.2 \ldots 1$ d)

$\Longrightarrow$ RR Lyr gap (change in color!).

Absolute magnitude of RR Lyr gap:

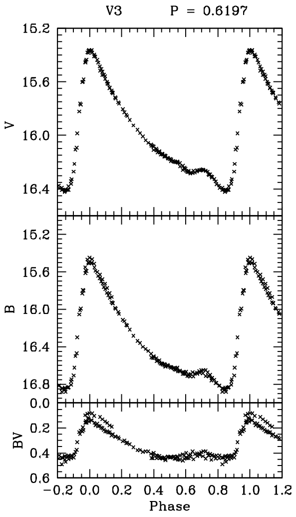
$M_V = 0.6$, $M_B = 0.8$ mag, i.e.,

$L_{RR} \sim 50\, L_\odot$.

M determined from ZAMS fitting, statistical parallax, and Baade-Wesselink method.

M2: Lee & Carney (1999, Fig. 2)

RR Lyr



Lightcurve shows characteristic color variations over pulsation (temperature change!), and a fast rise, slow decay behavior.

RR Lyr in GCs show bimodal number distribution due to a metallicity effect:

- RRab with $P > 0.5$ d and most probable period of $P_{ab} \sim 0.7$ d, and
- RRc, with $P < 0.5$ d and $P_c \sim 0.3$ d.

M is larger for higher Z, i.e., metal-rich RR Lyr are *fainter*

$\Longrightarrow$ difference in RR Lyr from population I and II.

RR Lyr work out to LMC and other dwarf galaxies of local group, however, used mainly for globular clusters.

(Lee & Carney, 1999, Fig. 5)

Interlude

Previous methods: Selection of methods for distances within Milky Way (and Magellanic Clouds): Basis for extragalactic distance scale.

Primary extragalactic distance indicators: Distance can be calibrated from observations *within* milky way or from theoretical grounds.

Primary indicators usually work within our neighborhood (i.e., out to Virgo cluster at 15–20 Mpc).

Examples: Cepheids, light echos,. . .

Secondary extragalactic distance indicators: Distance calibrated from primary distance indicators.

Examples: Type Ia SNe, methods based on integral galaxy properties.