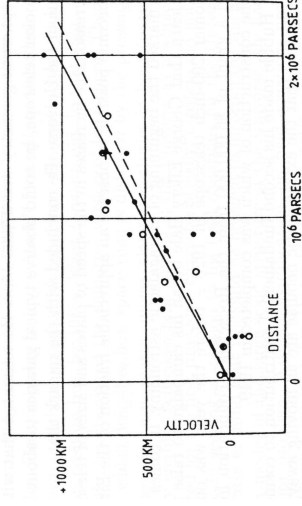


Cosmology – Basic Facts

Expansion, I



Hubble (1929): Velocity v
(defined as $v/c := z = \Delta\lambda/\lambda$)
for galaxy at distance r is

$$v(r) = H_0 r + v_X \cos \alpha \cos \delta + v_Y \sin \alpha \cos \delta + v_Z \sin \delta \quad (8.1)$$

(Hubble, 1929, Fig. 1)

(v_X, v_Y, v_Z) velocity due to motion of solar system ($\sim 350 \text{ km s}^{-1}$ towards $l = 264^\circ, b = 48^\circ$, Bennet et al., 1996)

H_0 : "Hubble parameter"; *intrinsic* component of velocity due to expansion of the universe.

Old usage: "Hubble constant", but $H_0 \neq \text{const.}$ (cf. Eq. (9.49)).

Basic Facts

Basic Facts

Cosmology deals with answering the questions about the universe as a whole.

The main question is:

How did the universe evolve into what it is now?

For this, *four major facts* need to be taken into account:

- expanding,
- isotropic,
- and homogeneous.

The isotropy and homogeneity of the universe is called the *cosmological principle*.

Perhaps (for us) the most important fact is:

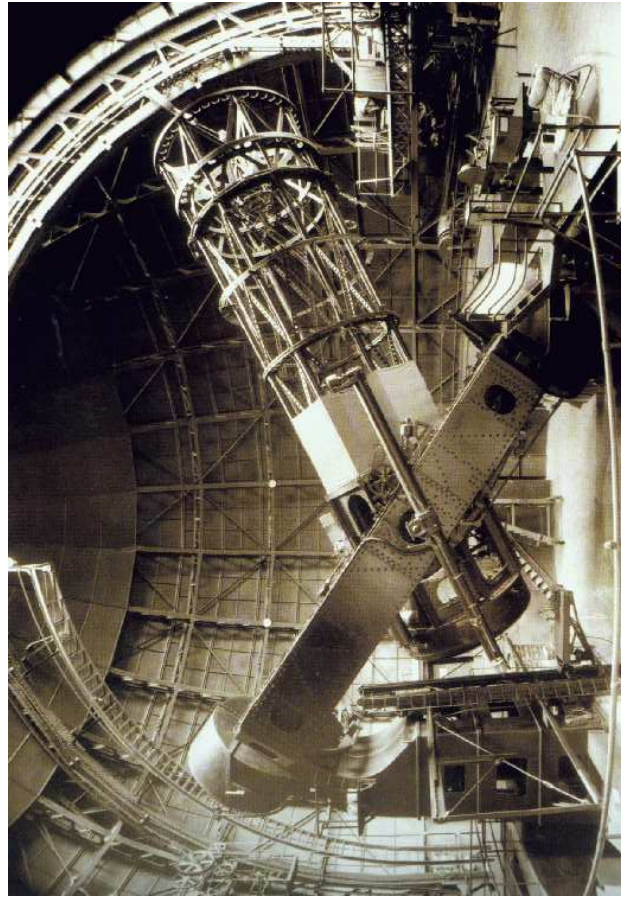
- The universe is *habitable to humans*.

i.e., the *anthropic principle*.

The one question cosmology does not attempt to answer is: How came the universe into being?

⇒ Realm of theology!

Basic Facts



"Hubble's" 2.5 m (100-inch) telescope on Mt. Wilson
(Image: http://sckim.kasi.re.kr/Images/hooker2_5m.gif)



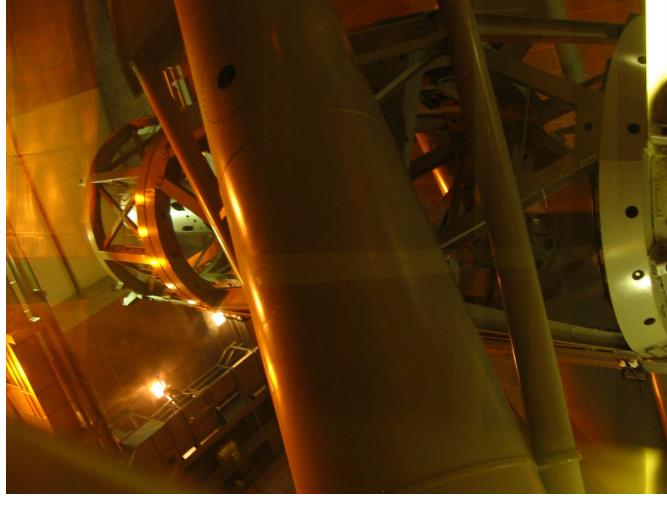
Dome of the 5 m Hale Telescope on Mt. Palomar (©I. Kreykenbohm)



The 5 m Hale Telescope (© I. Kreykenbohm)



Dome of the 5 m Hale Telescope on Mt. Palomar (© J. Wilms)



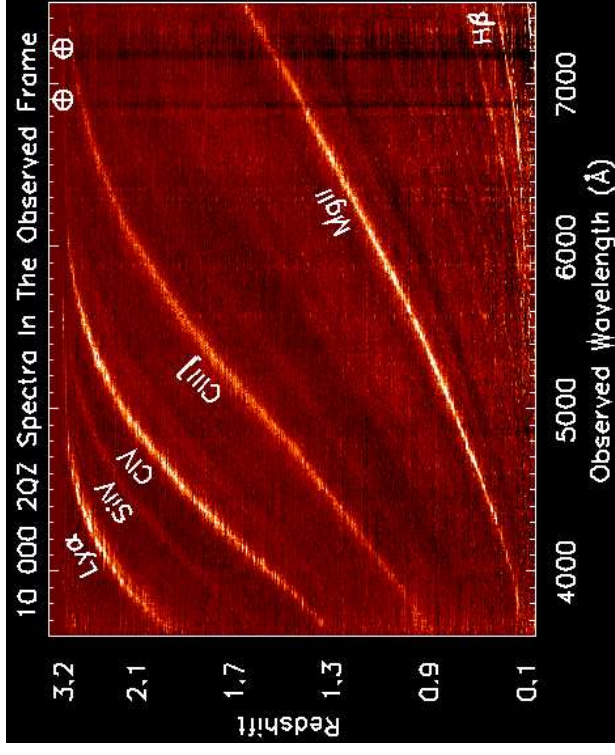
The 5 m Hale Telescope (© I. Kreykenbohm)



Mount of the 5 m Hale Telescope (© I. Kreykenbohm)

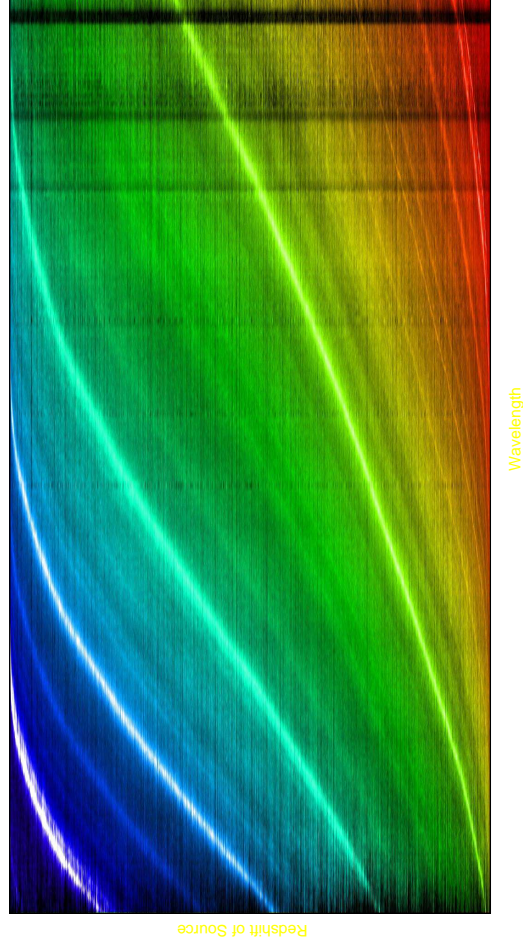


No comment (© I. Kreykenbohm)



courtesy 2dF QSO Redshift survey

As a consequence of the cosmological redshift, for different z different parts of the spectrum of a distant source are visible.





Expansion, XIII

Expansion law $v = H_0 r$ is unchanged under rotation and translation: isomorphism.

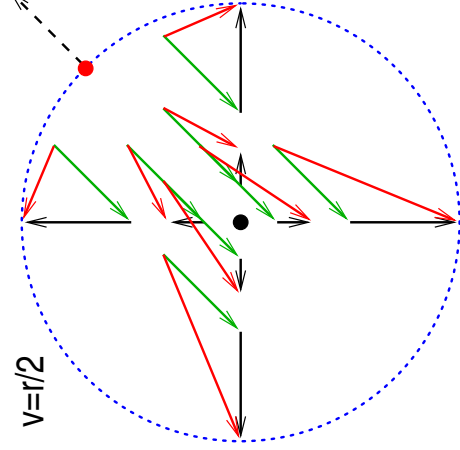
Proof:

Rotation: Trivial.

Translation: Observations from place with position r' and velocity v' : Observed distance is $r_0 = r - r'$, observed velocity is $v_0 = v - v'$. Because of the Hubble law,

$$v_0 = H_0 r - H_0 r' = H_0 (r - r') = H_0 r_0$$

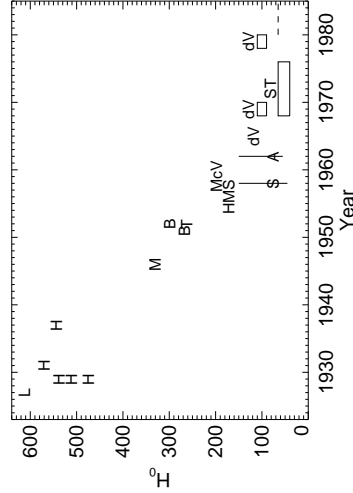
This isomorphism is a direct consequence of the homogeneity of the universe.



Despite everything receding from us, we are not at the center of the universe \implies Copernicus principle still holds.



Expansion, XII



Currently accepted value:

$$H_0 \sim 75 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

The systematic uncertainty of

$$H_0 \text{ is } \sim 10 \text{ km}^{-1} \text{ s}^{-1} \text{ Mpc}^{-1}$$

Parameterize uncertainty in formulae by defining

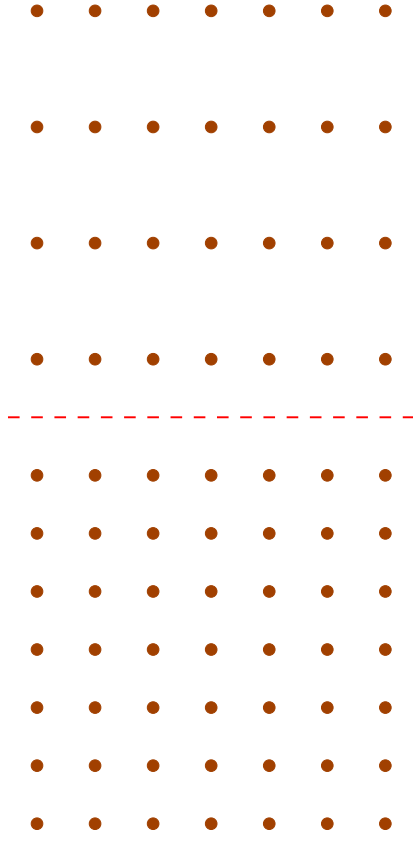
$$H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1} \cdot h$$

$$H_0 = 75 \text{ km s}^{-1} \text{ Mpc}^{-1} \cdot h_{75} \quad (8.2)$$

(after Trimble, 1997)

Note: H_0^{-1} has units of time: $H_0^{-1} = 9.78 \text{ Gyr}/h$; Hubble-Time; for $h = 0.75$, the Hubble-Time is 13 Gyr.

Homogeneity and Isotropy, I



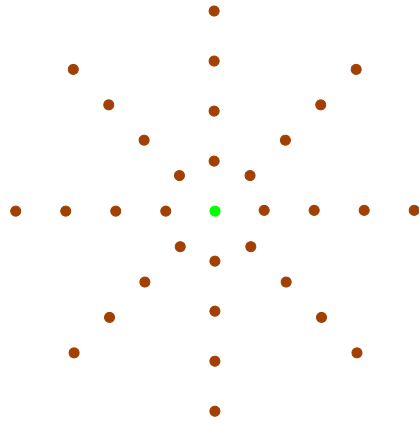
after Silk (1997, p. 8).

Note that homogeneity does not imply isotropy!



8-17

Homogeneity and Isotropy, II



Neither does isotropy *around one point* imply homogeneity!

⇒ Both assumptions need to be tested.

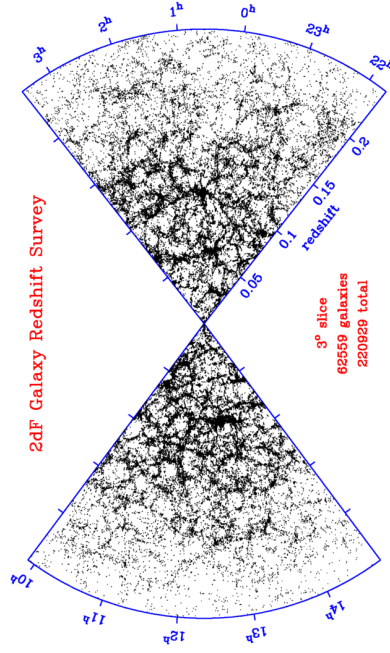
Basic Facts

16



8-18

Homogeneity, I



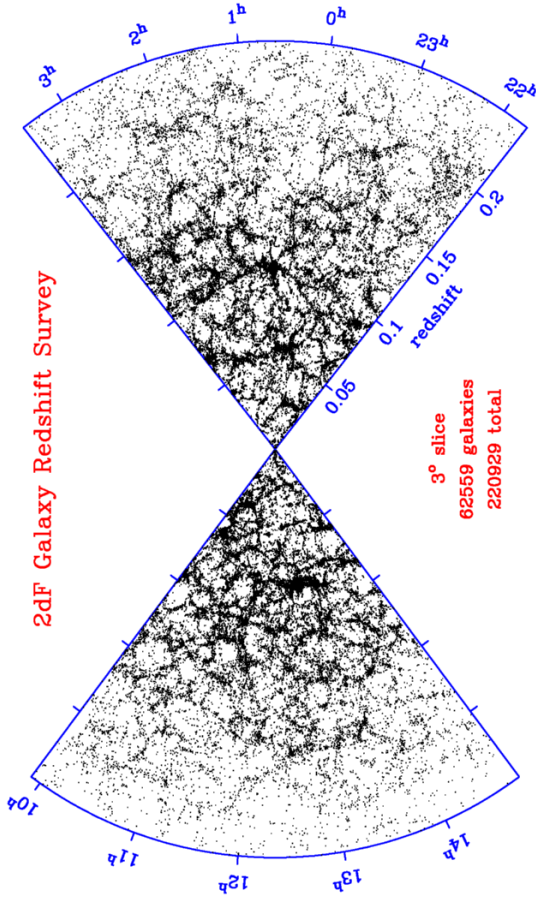
2dF Survey, ~220000 galaxies total

The universe is homogeneous ⇔ The universe looks the same everywhere in space

Testable by observing spatial distribution of galaxies.

Basic Facts

17

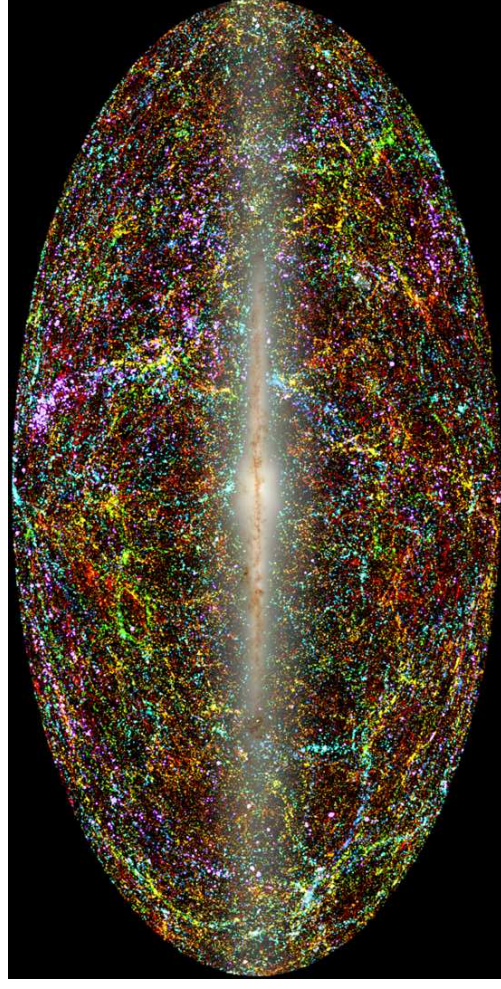


2dF Survey, ~220000 galaxies total

On scales $\gg 100$ Mpc the universe looks indeed the same.

Below that: structure.

Structures seen are galaxy clusters (gravitationally bound) and superclusters (larger structures, not [yet] gravitationally bound).

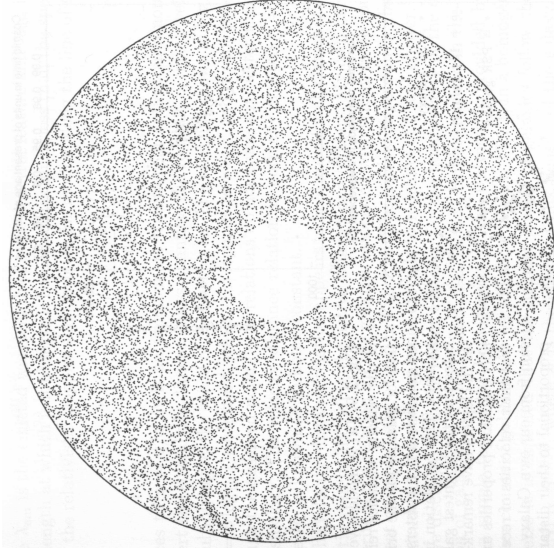


(Jarrett, 2004, Fig. 1)

Distribution of Galaxy redshifts in the 2MASS galaxy catalogue



Isotropy



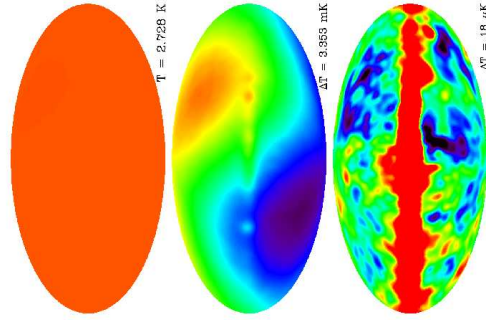
The universe is isotropic
 \iff The universe looks the same in all directions
 Radio galaxies are mainly quasars
 \implies Sample large space volume ($z \gtrsim 1$)
 \implies Clear isotropy.

Peebles (1993): Distribution of 31000 objects at $\lambda = 6$ cm from the Greenbank Catalogue.
 Anisotropy in the image: galactic plane, exclusion region around Cyg A, Cas A, and the north celestial pole.

Bennet, C. L., et al., 1996, *Apl*, 464, L1
 Hubble, E. P., 1929, 15, 168
 Jarrett, T., 2004, *Proc. Astron. Soc. Aust.*, 21, 396
 Peebles, P. J. E., 1983, *Principles of Physical Cosmology*, (Princeton: Princeton Univ. Press)
 Silk, J., 1997, *A Short History of the Universe*, Scientific American Library 53, (New York: W. H. Freeman)
 Trimble, V., 1997, *Space Sci. Rev.*, 79, 793



Isotropy



Best evidence for isotropy: Intensity of 3 K Cosmic Microwave Background (CMB) radiation.

First: dipole anisotropy due to motion of Sun (see slide 8-3), after subtraction: $\Delta T / T \lesssim 10^{-4}$ on scales from $10'$ to 180° .

At level of 10^{-5} : structure in CMB due to structure of surface of last scattering of the CMB photons, i.e., structure at the time when Hydrogen recombined.



World Models



Structure

Observations: cosmological principle holds: The universe is homogeneous and isotropic.

⇒ Need theoretical framework obeying the cosmological principle.

Use combination of

- General Relativity
- Thermodynamics
- Quantum Mechanics

⇒ Complicated!

For 99% of the work, the above points can be dealt with separately:

1. Define metric obeying cosmological principle.
2. Obtain equation for evolution of universe using Einstein field equations.
3. Use thermo/QM to obtain equation of state.
4. Solve equations.

Introduction

1



GRT vs. Newton

Before we can start to think about universe: Brief introduction to assumptions of general relativity.

⇒ See theory lectures for the gory details, or check with the literature (Weinberg or MTW).

Assumptions of GRT:

- Space is 4-dimensional, might be curved
- Matter (=Energy) modifies space (Einstein field equation).
- Covariance: physical laws must be formulated in a coordinate-system independent way.
- Strong equivalence principle: There is no experiment by which one can distinguish between free falling coordinate systems and inertial systems.
- At each point, space is locally Minkowski (i.e., locally, SRT holds).

⇒ Understanding of geometry of space necessary to understand physics.

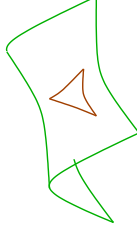
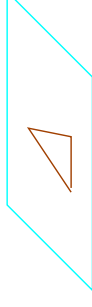
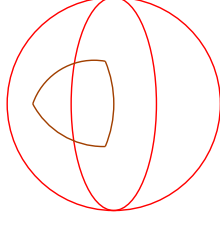
FRW Metric

1



2D Metrics

Before describing the 4D geometry of the universe: first look at 2D spaces (easier to visualize).



After Silk (1997, p. 107)

There are three classes of isotropic and homogeneous two-dimensional spaces:

- 2-sphere (\mathcal{S}^2) positively curved
- x - y -plane (\mathbb{R}^2) zero curvature
- hyperbolic plane (\mathcal{H}^2) negatively curved (curvature $\approx \sum$ angles in triangle $>$, $=$, or $<$ 180°)

We will now calculate what the metric for these spaces looks like.

FRW Metric

2



2D Metrics

The metric describes the local geometry of a space.

Differential distance, ds , in Euclidean space, \mathbb{R}^2 :

$$ds^2 = dx_1^2 + dx_2^2 \quad (9.1)$$

The metric tensor, $g_{\mu\nu}$, is defined through

$$ds^2 = \sum_{\mu} \sum_{\nu} g_{\mu\nu} dx^\mu dx^\nu \quad (9.2)$$

(Einstein's summation convention)

Thus, for the \mathbb{R}^2 ,

$$\begin{aligned} g_{11} &= 1 & g_{12} &= 0 \\ g_{21} &= 0 & g_{22} &= 1 \end{aligned} \quad (9.3)$$

FRW Metric

3

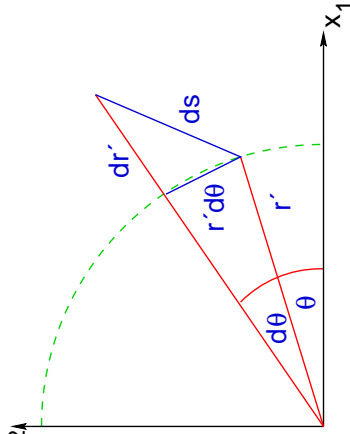


2D Metrics

But: Other coordinate-systems are also possible in the plane!

Changing to polar coordinates r', θ , defined by

x_2



$$x_1 =: r' \cos \theta \quad (9.4)$$

$$x_2 =: r' \sin \theta$$

it is easy to see that

$$ds^2 = dr'^2 + r'^2 d\theta^2 \quad (9.5)$$

Performing a change of scale by substituting $r' = Rr$, then gives

$$ds^2 = R^2 \{ dr^2 + r^2 d\theta^2 \} \quad (9.6)$$

FRW Metric

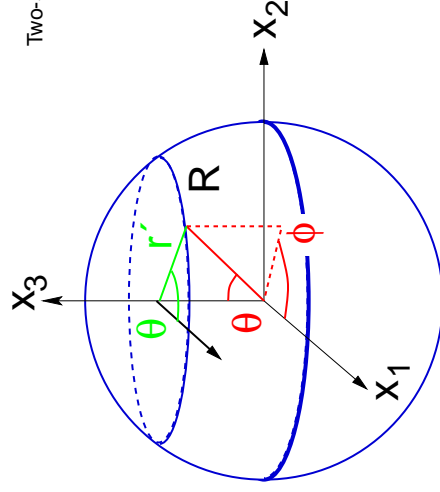
4



2D Metrics

A more complicated case occurs if **space is curved**.

Easiest case: surface of three-dimensional sphere (a two-sphere).



Two-sphere with radius R in \mathbb{R}^3 :

$$x_1^2 + x_2^2 + x_3^2 = R^2 \quad (9.7)$$

Length element of \mathbb{R}^3 :

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

Eq. (9.7) gives

$$x_3 = \sqrt{R^2 - x_1^2 - x_2^2}$$

such that

$$\begin{aligned} dx_3 &= \frac{\partial x_3}{\partial x_1} dx_1 + \frac{\partial x_3}{\partial x_2} dx_2 \\ &= -\frac{x_1 dx_1 + x_2 dx_2}{\sqrt{R^2 - x_1^2 - x_2^2}} \end{aligned} \quad (9.8)$$

After Kolb & Turner (1990, Fig. 2.1)

FRW Metric

5



2D Metrics

Introduce again polar coordinates r', θ in x_3 -plane:

$$x_1 =: r' \cos \theta, x_2 =: r' \sin \theta \quad (9.4)$$

(note: r', θ are only unique in upper or lower half-sphere)

The differentials are given by

$$dx_1 = \cos \theta dr' - r' \sin \theta d\theta \quad \text{and} \quad dx_2 = \sin \theta dr' + r' \cos \theta d\theta \quad (9.9)$$

In cartesian coordinates, the length element on \mathcal{S}^2 is

$$ds^2 = dx_1^2 + dx_2^2 + \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2} \quad (9.10)$$

inserting eq. (9.9) gives after some algebra

$$= r'^2 d\theta^2 + \frac{R^2}{R^2 - r'^2} dr'^2 \quad (9.11)$$

finally, defining $r = r'/R$ (i.e., $0 \leq r \leq 1$) results in

$$ds^2 = R^2 \left\{ \frac{dr^2}{1-r^2} + r^2 d\theta^2 \right\} \quad (9.12)$$

FRW Metric

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2D Metrics

Alternatively, we can work in spherical coordinates on \mathcal{S}^2

$$\begin{aligned} x_1 &= R \sin \theta \cos \phi \\ x_2 &= R \sin \theta \sin \phi \\ x_3 &= R \cos \theta \end{aligned} \quad (9.13)$$

($\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$).

Going through the same steps as before, we obtain after some tedious algebra

$$ds^2 = R^2 \{ d\theta^2 + \sin^2 \theta d\phi^2 \} \quad (9.14)$$

FRW Metric

7



2D Metrics

(Important) remarks:

1. The 2-sphere has no edges, has no boundaries, but has still a finite volume, $V = 4\pi R^2$.
2. Expansion or contraction of sphere caused by variation of $R \implies R$ determines the scale of volumes and distances on \mathcal{S}^2 .

R is called the **scale factor**

3. Positions on \mathcal{S}^2 are defined, e.g., by r and θ , independent on the value of R

r and θ are called **comoving coordinates**

4. Although the metrics Eq. (9.10), (9.12), and (9.14) look very different, they still describe the same space \implies that's why physics should be covariant, i.e., independent of the coordinate system!

FRW Metric

8



2D Metrics

The hyperbolic plane, \mathcal{H}^2 , is defined by

$$x_1^2 + x_2^2 - x_3^2 = -R^2 \quad (9.15)$$

If we work in **Minkowski space**, where

$$ds^2 = dx_1^2 + dx_2^2 - dx_3^2 \quad (9.16)$$

then

$$= dx_1^2 + dx_2^2 - \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 + x_1^2 + x_2^2} \quad (9.17)$$

\implies substitute $R \rightarrow iR$ (where $i = \sqrt{-1}$) to obtain same form as for sphere (eq. 9.11)!

Therefore,

$$ds^2 = R^2 \left\{ \frac{dr^2}{1+r^2} + r^2 d\theta^2 \right\} \quad (9.18)$$

FRW Metric

9



2D Metrics

The analogy to spherical coordinates on the hyperbolic plane are given by

$$\begin{aligned} x_1 &= R \sinh \theta \cos \phi \\ x_2 &= R \sinh \theta \sin \phi \\ x_3 &= R \cosh \theta \end{aligned} \quad (9.19)$$

$$(\theta \in [-\infty, +\infty], \phi \in [0, 2\pi]).$$

A session with Maple (see handout) will convince you that these coordinates give

$$ds^2 = R^2 \{ d\theta^2 + \sinh^2 \theta d\phi^2 \} \quad (9.20)$$

Remark:

\mathcal{H}^2 is unbound and has an infinite volume.

FRW Metric

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9-12

Transcript of Maple session to obtain Eq. (9.20):

```
> x1:=r*sinh(theta)*cos(phi);
  x1 := r sinh(theta) cos(phi)
> x2:=r*sinh(theta)*sin(phi);
  x2 := r sinh(theta) sin(phi)
> x3:=r*cosh(theta);
  x3 := r cosh(theta)
> dx1:=diff(x1,theta)*dtheta+diff(x1,phi)*dphi;
  dx1 := r cosh(theta) cos(phi) dtheta - r sinh(theta) sin(phi) dphi
> dx2:=diff(x2,theta)*dtheta+diff(x2,phi)*dphi;
  dx2 := r cosh(theta) sin(phi) dtheta + r sinh(theta) cos(phi) dphi
> ds2:=dx1^2+dx2^2-dx3^2-(x1^2+dx1^2+dx2^2)^2/(r^2+x1^2+x2^2);
  ds2 := (r cosh(theta) cos(phi) dtheta - r sinh(theta) sin(phi) dphi)^2
+ (r cosh(theta) sin(phi) dtheta + r sinh(theta) cos(phi) dphi)^2 - (
r sinh(theta) cos(phi) (r cosh(theta) cos(phi) dtheta - r sinh(theta) sin(phi) dphi)
+ r sinh(theta) sin(phi) (r cosh(theta) sin(phi) dtheta + r sinh(theta) cos(phi) dphi))^2 / (
r^2 + r^2 sinh(theta)^2 cos(phi)^2 + r^2 sinh(theta)^2 sin(phi)^2)
> expand(ds2);
r^2 cosh(theta)^2 cos(phi)^2 dtheta^2 + r^2 sinh(theta)^2 sin(phi)^2 dphi^2 + r^2 cosh(theta)^2 sin(phi)^2 dtheta^2
+ r^2 sinh(theta)^2 cos(phi)^2 dphi^2 - r^4 sinh(theta)^2 cos(phi)^4 cosh(theta)^2 dtheta^2
- r^4 sinh(theta)^2 cos(phi)^2 cosh(theta)^2 dtheta^2 sin(phi)^2 - r^4 sinh(theta)^2 sin(phi)^4 cosh(theta)^2 dtheta^2
%1
%1 := r^2 + r^2 sinh(theta)^2 cos(phi)^2 + r^2 sinh(theta)^2 sin(phi)^2
> simplify("",{cosh(theta)^2-sinh(theta)^2=1},[cosh(theta)]);
r^2 dtheta^2 + r^2 sinh(theta)^2 dphi^2
```

**2D Metrics**

To summarize:

Sphere:
$$ds^2 = R^2 \left\{ \frac{dr^2}{1-r^2} + r^2 d\theta^2 \right\} \quad (9.12)$$

Plane:
$$ds^2 = R^2 \left\{ dr^2 + r^2 d\theta^2 \right\} \quad (9.6)$$

Hyperbolic Plane:
$$ds^2 = R^2 \left\{ \frac{dr^2}{1+r^2} + r^2 d\theta^2 \right\} \quad (9.18)$$

⇒ All three metrics can be written as

$$ds^2 = R^2 \left\{ \frac{dr^2}{1-k \cdot r^2} + r^2 d\theta^2 \right\} \quad (9.21)$$

where k : defines the geometry:

$$k = \begin{cases} +1 & \text{spherical} \\ 0 & \text{planar} \\ -1 & \text{hyperbolic} \end{cases} \quad (9.22)$$

FRW Metric

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**2D Metrics**

For "spherical coordinates" we found:

Sphere:
$$ds^2 = R^2 \left\{ d\theta^2 + \sin^2 \theta d\phi^2 \right\} \quad (9.14)$$

Plane:
$$ds^2 = R^2 \left\{ d\theta^2 + \theta^2 d\phi^2 \right\} \quad (9.6)$$

Hyperbolic Plane:
$$ds^2 = R^2 \left\{ d\theta^2 + \sinh^2 \theta d\phi^2 \right\} \quad (9.20)$$

⇒ All three metrics can be written as

$$ds^2 = R^2 \left\{ d\theta^2 + S_k^2(\theta) d\phi^2 \right\} \quad (9.23)$$

where

$$S_k(\theta) = \begin{cases} \sin \theta & \text{for } k = +1 \\ \theta & \text{for } k = 0 \\ \sinh \theta & \text{for } k = -1 \end{cases} \quad \text{and} \quad C_k(\theta) = \begin{cases} \cos \theta & \text{for } k = +1 \\ 1 & \text{for } k = 0 \\ \cosh \theta & \text{for } k = -1 \end{cases} \quad (9.24)$$

The cos-like analogue of S_k, C_k , will be needed later

Note that, compared to the earlier formulae, some coordinates have been renamed. This is confusing, but let's gal...

FRW Metric

12

**RW Metric**

• Cosmological principle + expansion ⇒ ∃ freely expanding cosmical coordinate system.

- Observers =: fundamental observers
- Time =: cosmic time

This is the coordinate system in which the 3K radiation is isotropic, clocks can be synchronized, e.g., by adjusting time to the local density of the universe.

⇒ Metric has temporal and spatial part.

This also follows directly from the equivalence principle.

• Homogeneity and isotropy ⇒ spatial part is spherically symmetric:

$$d\psi^2 := d\theta^2 + \sin^2 \theta d\phi^2 \quad (9.25)$$

• Expansion: ∃ scale factor, $R(t)$ ⇒ measure distances using comoving coordinates.

⇒ metric looks like

$$ds^2 = c^2 dt^2 - R^2(t) \left[f^2(r) dr^2 + g^2(r) d\psi^2 \right] \quad (9.26)$$

where $f(r)$ and $g(r)$ are arbitrary.

FRW Metric

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**RW Metric**

Metrics of the form of eq. (9.26) are called Robertson-Walker (RW) metrics (introduced in 1935).

Previously studied by Friedmann and Lemaitre...

One common choice is

$$ds^2 = c^2 dt^2 - R^2(t) \left[dr^2 + S_k^2(r) d\psi^2 \right] \quad (9.27)$$

where

$R(t)$: scale factor, containing the physics

t : cosmic time

r, θ, ϕ : comoving coordinates (remember Eq. (9.25) ($d\psi^2 := d\theta^2 + \sin^2 \theta d\phi^2$!))

k : defines curvature, integer

$S_k(r)$ was defined in Eq. (9.24).

Remark: θ and ϕ describe directions on sky, as seen from the arbitrary center of the coordinate system (=us), r can be interpreted as a radial coordinate.

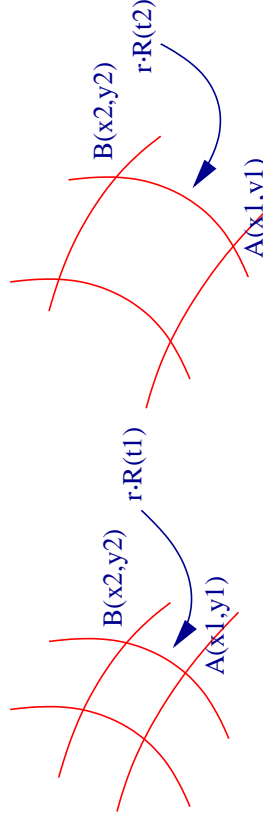
FRW Metric

14



RW Metric

The RW metric defines an universal coordinate system tied to expansion of space:



Scale factor $R(t)$ describes evolution of universe.

- r is called the comoving distance.
- $D(t) := r \cdot R(t)$ is called the proper distance, (e.g., $r \cdot R(t)$ is measured in Mpc)

FRW Metric

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RW Metric

Other forms of the RW metric are also used:

1. Substitution $S_k(r) \rightarrow r$ gives

$$ds^2 = c^2 dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\psi^2 \right\} \quad (9.28)$$

(i.e., other definition of comoving radius r , which is still dimensionless).

2. A metric with a dimensionless scale factor,

$$a(t) := \frac{R(t)}{R(t_0)} = \frac{R(t)}{R_0} \quad (9.29)$$

(where $t_0 = \text{today}$, i.e., $a(t_0) = 1$), gives

$$ds^2 = c^2 dt^2 - a^2(t) \left\{ dr^2 + \frac{S_k^2(R_0 r)}{R_0^2} d\psi^2 \right\} \quad (9.30)$$

FRW Metric

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RW Metric

3. Using $a(t)$ and the substitution $S_k(r) \rightarrow r$ is also possible:

$$ds^2 = c^2 dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - k \cdot (R_0 r)^2} + r^2 d\psi^2 \right\} \quad (9.31)$$

The units of $R_0 r$ are Mpc \Rightarrow Used for observations!

4. Replace cosmic time, t , by conformal time, $d\eta = dt/R(t)$ \Rightarrow conformal metric,

$$ds^2 = R^2(\eta) \left\{ d\eta^2 - \frac{dr^2}{1 - kr} - r^2 d\psi^2 \right\} \quad (9.32)$$

Theoretical importance of this metric: For $k = 0$, i.e., a flat space, the RW metric = Minkowski line element $\times R^2(\eta) \Rightarrow$ Equivalence principle!

FRW Metric

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RW Metric

5. Finally, the metric can also be written in the isotropic form,

$$ds^2 = c^2 dt^2 - \frac{R(t)}{1 + (k/4)r^2} \{ dr^2 + r^2 d\psi^2 \} \quad (9.33)$$

Here, the term in $\{ \dots \}$ is just the line element of a 3d-sphere \Rightarrow isotropy!

Note: There are as many notations as authors, e.g., some use $a(t)$ where we use $R(t)$, etc. \Rightarrow Be careful!

Note 2: Local homogeneity and isotropy (i.e., within a Hubble radius, $r = c/H_0$), do not imply global homogeneity and isotropy \Rightarrow Cosmologies with a non-trivial topology are possible (e.g., also with more dimensions. ...).

FRW Metric

18



Friedmann Equations, I

General relativistic approach: Insert metric into Einstein equation to obtain differential equation for $R(t)$:

Einstein equation:

$$R_{\mu\nu} - \underbrace{\frac{1}{2}Rg_{\mu\nu}}_{G_{\mu\nu}} = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \quad (9.34)$$

where

$g_{\mu\nu}$: Metric tensor ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$)

$R_{\mu\nu}$: Ricci tensor (function of $g_{\mu\nu}$)

R : Ricci scalar (function of $g_{\mu\nu}$)

$G_{\mu\nu}$: Einstein tensor (function of $g_{\mu\nu}$)

$T_{\mu\nu}$: Stress-energy tensor, describing curvature of space due to fields present (matter, radiation, ...)

Λ : Cosmological constant

⇒ Messy, but doable

Dynamics

1



Friedmann Equations, II

Here, Newtonian derivation of Friedmann equations: Dynamics of a mass element on the surface of sphere of density $\rho(t)$ and comoving radius d , i.e., proper radius $d \cdot R(t)$ (McCrea, 1937)

Mass of sphere:

$$M = \frac{4\pi}{3}(dR)^3 \rho(t) = \frac{4\pi}{3}d^3 \rho_0 \quad \text{where} \quad \rho(t) = \frac{\rho_0}{R(t)^3} \quad (9.35)$$

Force on mass element:

$$m \frac{d^2}{dt^2}(dR(t)) = -\frac{GMm}{(dR(t))^2} = -\frac{4\pi G}{3} \frac{d\rho_0}{R^2(t)} m \quad (9.36)$$

Canceling $m \cdot d$ gives momentum equation:

$$\ddot{R}(t) = -\frac{4\pi G}{3} \frac{\rho_0}{R(t)^2} = -\frac{4\pi G}{3} \rho(t) R(t) \quad (9.37)$$

Multiplying Eq. (9.37) with \dot{R} and integrating yields the energy equation:

$$\frac{1}{2}\dot{R}(t)^2 = +\frac{4\pi G}{3} \frac{\rho_0}{R(t)} + \text{const.} = +\frac{4\pi G}{3} \rho(t) R^2(t) + \text{const.} \quad (9.38)$$

where the constant can only be obtained from GR.

Dynamics

2



Friedmann Equations, III

Problems with the Newtonian derivation:

1. Cloud is implicitly assumed to have $r_{\text{cloud}} < \infty$

(for $r_{\text{cloud}} \rightarrow \infty$ the force is undefined)

⇒ violates cosmological principle.

2. Particles move *through* space

⇒ $v > c$ possible

⇒ violates SRT.

Why do we get correct result?

GRT → Newton for small scales and mass densities

Since universe is isotropic: scale invariance on Mpc scales

⇒ Newton sufficient (classical limit of GR).

(In fact, point 1 above does hold in GR: Birkhoff's theorem).

Dynamics

3



Friedmann Equations, IV

The exact GR derivation of Friedmanns equation gives:

$$\begin{aligned} \ddot{R} &= -\frac{4\pi G}{3}R \left(\rho + \frac{3p}{c^2} \right) + \left[\frac{1}{3}\Lambda R \right] \\ \dot{R}^2 &= +\frac{8\pi G\rho}{3}R^2 - kc^2 + \left[\frac{1}{3}\Lambda c^2 R^2 \right] \end{aligned} \quad (9.39)$$

Notes:

1. For $k = 0$: Eq. (9.39) → Eq. (9.38).

2. k determines the curvature of space (and is *not* an integer here!).

3. The density, ρ , includes the contribution of all different kinds of energy (remember mass-energy equivalence!).

4. There is energy associated with the vacuum, parameterized by the parameter Λ .

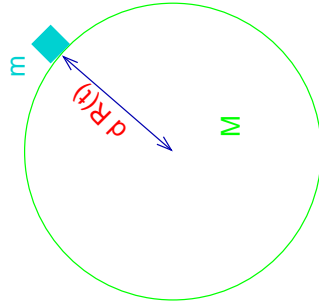
In Eq. 9.49, it will be shown that the Hubble parameter can be expressed as $H(t) = \frac{\dot{R}}{R}$.

For $\Lambda = 0$, it evolves as:

$$\left(\frac{\dot{R}}{R} \right)^2 = H^2(t) = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} \quad (9.40)$$

Dynamics

4



**The Critical Density, I**

Solving Eq. (9.40) for k :

$$\frac{R^2}{c} \left(\frac{8\pi G}{3} \rho - H^2 \right) = k \quad (9.41)$$

\implies Sign of curvature parameter k only depends on density, ρ . With

$$\rho_c = \frac{3H^2}{8\pi G} \quad \text{and} \quad \Omega = \frac{\rho}{\rho_c} \quad (9.42)$$

$\Omega > 1 \implies k > 0 \implies$ closed universe
 $\Omega = 1 \implies k = 0 \implies$ flat universe
 $\Omega < 1 \implies k < 0 \implies$ open universe

ρ_c is called the critical density.

For $\Omega \leq 1$ the universe will expand until ∞ ,

For $\Omega > 1$ we will see the “big crunch”.

Current value of ρ_c : $\sim 1.67 \times 10^{-24} \text{ g cm}^{-3}$ (3... 10 H-atoms m^{-3}).

Dynamics

**The Critical Density, II**

Ω has a second order effect on the expansion:

Taylor series of $R(t)$ around $t = t_0$:

$$\frac{R(t)}{R(t_0)} = \frac{R(t_0)}{R(t_0)} + \frac{\dot{R}(t_0)}{R(t_0)} (t - t_0) + \frac{1}{2} \frac{\ddot{R}(t_0)}{R(t_0)} (t - t_0)^2 \quad (9.43)$$

The Friedmann equation Eq. (9.37) can be written

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \rho = -\frac{4\pi G}{3} \Omega \frac{3H^2}{8\pi G} = -\frac{\Omega H^2}{2} \quad (9.44)$$

Since $H(t) = \dot{R}/R$ (Eq. 9.49), Eq. (9.43) is

$$\frac{R(t)}{R(t_0)} = 1 + H_0 (t - t_0) - \frac{1}{2} \frac{\Omega_0}{2} H_0^2 (t - t_0)^2 \quad (9.45)$$

where $H_0 = H(t_0)$ and $\Omega_0 = \Omega(t_0)$.

The subscript 0 is often omitted in the case of Ω .

Often, Eq. (9.45) is written using the deceleration parameter:

$$q := -\frac{\Omega}{2} = -\frac{\ddot{R}(t_0)R(t_0)}{R^2(t_0)} \quad (9.46)$$

Dynamics

**Hubble's Law**

Hubble's Law follows from the variation of $R(t)$:



Small scales \implies Euclidean geometry. Then the proper distance between two observers is:

$$D(t) = d \cdot R(t) \quad (9.47)$$

where d : comoving distance.

Expansion \implies proper separation changes:

$$\frac{\Delta D}{\Delta t} = \frac{R(t + \Delta t)d - R(t)d}{\Delta t} \implies \lim_{\Delta t \rightarrow 0} \implies v = \frac{dD}{dt} = \dot{R}d = \frac{\dot{R}}{R}D =: HD \quad (9.48)$$

\implies Identify local Hubble “constant” as

$$H = \frac{\dot{R}}{R} = \dot{a}(t) \quad (a(t) \text{ from Eq. 9.29, } a(\text{today}) = 1) \quad (9.49)$$

Since $R = R(t) \implies H$ is time-dependent!

Dynamics

**Redshift, I**

The cosmological redshift is a consequence of the expansion of the universe:

The comoving distance is constant, thus in terms of the proper distance:

$$d = \frac{D(t = \text{today})}{R(t = \text{today})} = \frac{D(t)}{R(t)} = \text{const.} \quad (9.50)$$

Set $a(t) = R(t)/R(t = \text{today})$, then eq. (9.50) implies

$$\lambda_{\text{obs}} = \frac{\lambda_{\text{emit}}}{a_{\text{emit}}} \quad (9.51)$$

(λ_{obs} : observed wavelength, λ_{emit} : emitted wavelength)

Thus the observed redshift is

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} - 1 = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} - 1 \quad (9.52)$$

$$\implies 1 + z = \frac{1}{a_{\text{emit}}} = \frac{R(t = \text{today})}{R(t)} = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} \quad (9.53)$$

Light emitted at $z = 1$ was emitted when the universe was half as big as today!

z : measure for relative size of universe at time the observed light was emitted.

Dynamics



Redshift, II

Outside of the local universe: Eq. (9.53) only valid interpretation of z .
 \implies It is common to interpret z as in special relativity:

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{9.61}$$

THIS IS WRONG

Redshift is due to expansion of space, not due to motion of galaxy.

What is true is that z is accumulation of many infinitesimal red-shifts à la Eq. (9.54), see, e.g., Peacock (1999).

Dynamics



Time Dilation

For light, $D = c \Delta t$. Then a consequence of Eq. (9.50) is

$$\frac{c \Delta t_{\text{emit}}}{R(t_{\text{emit}})} = \frac{c \Delta t_{\text{obs}}}{R(t_{\text{obs}})} \implies \frac{dt}{R} = \text{const.} \tag{9.59}$$

In other words:

$$\frac{dt_{\text{obs}}}{dt_{\text{emit}}} = \frac{R(t_{\text{obs}})}{R(t_{\text{emit}})} = 1 + z \tag{9.62}$$

\implies Time dilation of events at large z .

This cosmological time dilation has been observed in the light curves of supernova outbursts.

All other observables apart from z (e.g., number density $N(z)$, luminosity distance d_L , etc.) require explicit knowledge of $R(t)$

\implies Need to look at the dynamics of the universe.

Dynamics

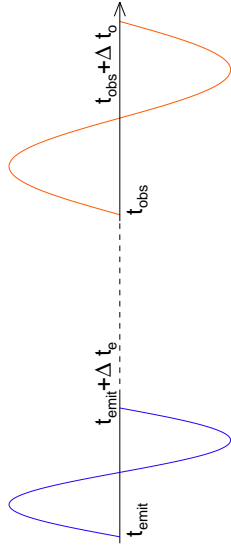
Note that the definition of H allows us to derive Hubble's relation for the case of small v , i.e., $v \ll c$. In this case, the red-shift is

$$z = \frac{v}{c} \implies z = \frac{Hd}{c} \tag{9.54}$$

An alternative derivation of the cosmological redshift follows directly from general relativity, using the basic GR fact that for photons $ds^2 = 0$. Inserting this into the metric, and assuming without loss of generality that $ds^2 = 0$, one finds

$$0 = c^2 dt^2 - R^2(t) dr^2 \implies dr = \pm \frac{c dt}{R(t)} \tag{9.55}$$

Since photons travel forward, we choose the + sign.



The comoving distance traveled by photons emitted at cosmic times t_{emit} and $t_{\text{emit}} + \Delta t_e$ is

$$r_1 = \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{c dt}{R(t)} \quad \text{and} \quad r_2 = \int_{t_{\text{emit}} + \Delta t_e}^{t_{\text{obs}} + \Delta t_0} \frac{c dt}{R(t)} \tag{9.56}$$

But the comoving distances are equal, $r_1 = r_2$. Therefore

$$0 = \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{c dt}{R(t)} - \int_{t_{\text{emit}} + \Delta t_e}^{t_{\text{obs}} + \Delta t_0} \frac{c dt}{R(t)} \tag{9.57}$$

$$= \int_{t_{\text{emit}} + \Delta t_e}^{t_{\text{obs}}} \frac{c dt}{R(t)} - \int_{t_{\text{obs}}}^{t_{\text{obs}} + \Delta t_0} \frac{c dt}{R(t)} \tag{9.58}$$

If Δt small $\implies R(t) \approx \text{const.}$:

$$= \frac{c \Delta t_e}{R(t_{\text{emit}})} - \frac{c \Delta t_0}{R(t_{\text{obs}})} \tag{9.59}$$

For a wave: $c \Delta t = \lambda$, such that

$$\frac{\lambda_{\text{emit}}}{R(t_{\text{emit}})} = \frac{\lambda_{\text{obs}}}{R(t_{\text{obs}})} \iff \frac{\lambda_{\text{emit}}}{\lambda_{\text{obs}}} = \frac{R(t_{\text{emit}})}{R(t_{\text{obs}})} \tag{9.60}$$

From this equation it is straightforward to derive Eq. (9.52).

**Equation of state, I**

Evolution of the universe determined by three different kinds of equation of state:

1. Matter: Normal (nonrelativistic) particles get diluted by expansion of the universe:

$$\rho_m \propto R^{-3} \quad (9.63)$$

Matter is also often called dust by cosmologists.
2. Radiation: The energy density of radiation decreases because of volume expansion and because of the cosmological redshift (Eq. 9.60): $\lambda_{\text{obs}}/\lambda_{\text{emit}} = \nu_{\text{emit}}/\nu_{\text{obs}} = R(t_{\text{obs}})/R(t_{\text{emit}})$ such that

$$\rho_r \propto R^{-4} \quad (9.64)$$
3. Vacuum: The vacuum energy density ($=\Lambda$) is independent of R :

$$\rho_v = \text{const.} \quad (9.65)$$

Inserting these equations of state into the Friedmann equation and solving with the boundary condition $R(t=0) = 0$ then gives a specific world model.

Dynamics

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**Equation of state, II**

Current scale factor is determined by H_0 and Ω_0 :

Friedmann for $t = t_0$:

$$\dot{R}_0^2 - \frac{8\pi G}{3} \rho R_0^2 = -k c^2 \quad (9.66)$$

Insert Ω and note $H_0 = \dot{R}_0/R_0$

$$\iff H_0^2 R_0^2 - H_0^2 \Omega_0 R_0^2 = -k c^2 \quad (9.67)$$

And therefore

$$R_0 = \frac{c}{H_0} \sqrt{\frac{k}{\Omega - 1}} \quad (9.68)$$

For $\Omega \rightarrow 0$, $R_0 \rightarrow c/H_0$, the Hubble length.

For $\Omega = 1$, R_0 is arbitrary.

We now have everything we need to solve the Friedmann equation and determine the evolution of the universe for $k = 0, +1, \text{ and } -1$.

Dynamics

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 **$k = 0$, Matter dominated**

For the matter dominated, flat case (the Einstein-de Sitter case), the Friedmann equation is

$$\dot{R}^2 - \frac{8\pi G}{3} \rho_0 \frac{R_0^3}{R^3} R^2 = 0 \quad (9.69)$$

For $k = 0$: $\Omega = 1$ and

$$\frac{8\pi G \rho_0}{3} = \Omega_0 H_0^2 R_0^3 = H_0^2 R_0^3 \quad (9.70)$$

Therefore, the Friedmann eq. is

$$\dot{R}^2 - \frac{H_0^2 R_0^3}{R} = 0 \implies \frac{dR}{dt} = H_0 R_0^{3/2} R^{-1/2} \quad (9.71)$$

Separation of variables and setting $R(0) = 0$,

$$\int_0^{R(t)} R^{1/2} dR = H_0 R_0^{3/2} t \implies \frac{2}{3} R^{3/2}(t) = H_0 R_0^{3/2} t \implies R(t) = R_0 \left(\frac{3H_0 t}{2} \right)^{2/3} \quad (9.72)$$

Therefore, for $k = 0$, the universe expands until ∞ , its current age ($R(t_0) = R_0$) is given by

$$t_0 = \frac{2}{3H_0} \quad (9.73)$$

Reminder: The Hubble-Time is $H_0^{-1} = 9.78 \text{ G}_{\text{yr}}/h$.

Dynamics

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9-33

For the matter dominated, closed case, Friedmann's equation is

$$\dot{R}^2 - \frac{8\pi G}{3} \rho_0 \frac{R_0^3}{R} = -c^2 \iff \dot{R}^2 - \frac{H_0^2 R_0^3}{R} = -c^2 \quad (9.74)$$

Inserting R_0 from Eq. (9.68) gives

$$\dot{R}^2 - \frac{H_0^2 c^3 \Omega_0}{H_0^3 (\Omega_0 - 1)^{3/2}} \frac{1}{R} = -c^2 \quad (9.75)$$

which is equivalent to

$$\frac{dR}{dt} = c \left(\frac{\xi}{R} - 1 \right)^{1/2} \quad \text{with } \xi = \frac{c}{H_0 (\Omega_0 - 1)^{3/2}} \quad (9.76)$$

With the boundary condition $R(0) = 0$, separation of variables gives

$$ct = \int_0^{R(t)} \frac{dR}{(\xi/R - 1)^{1/2}} = \int_0^{R(t)} \frac{\sqrt{R} dR}{(\xi - R)^{1/2}} \quad (9.77)$$

Integration by substitution gives the 'cycloid solution'

$$R = \xi \sin^2 \frac{\theta}{2} = \frac{\xi}{2} (1 - \cos \theta) \quad \text{and} \quad ct = \frac{\xi}{2} (\theta - \sin \theta) \quad (9.78)$$

where θ is an implicit parameter.

The age of the universe, t_0 , is obtained by solving

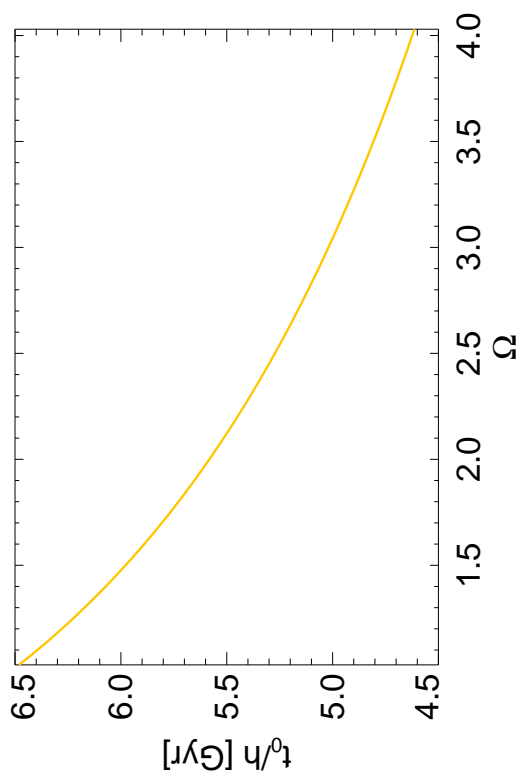
$$R_0 = \frac{c}{H_0 (\Omega_0 - 1)^{1/2}} = \frac{\xi}{2} (1 - \cos \theta_0) = \frac{c}{2 H_0 (\Omega_0 - 1)^{1/2}} (1 - \cos \theta_0) \quad (9.79)$$

(remember Eq. 9.68). Therefore

$$\cos \theta_0 = 2 - \frac{\Omega_0}{\Omega_0 - 1} \iff \sin \theta_0 = \frac{2}{\Omega_0} \sqrt{\Omega_0 - 1} \quad (9.80)$$

Inserting this into Eq. (9.78) gives

$$t_0 = \frac{1}{2H_0 (\Omega_0 - 1)^{3/2}} \left[\arccos \left(\frac{2 - \Omega_0}{\Omega_0} \right) - \frac{2}{\Omega_0} \sqrt{\Omega_0 - 1} \right] \quad (9.81)$$

 **$k = +1$, Matter dominated, II**

Age of a closed and matter dominated universe.

Dynamics

 **$k = -1$, Matter dominated, I**

Finally, the matter dominated, open case. This case is very similar to the case of $k = +1$:

For $k = -1$, the Friedmann equation becomes

$$\frac{dR}{dt} = c \left(\frac{\zeta}{R} + 1 \right)^{1/2} \quad (9.84)$$

where

$$\zeta = \frac{c}{H_0} \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} \quad (9.85)$$

Separation of variables gives after a little bit of algebra

$$R = \frac{\zeta}{2} (\cosh \theta - 1) \quad (9.86)$$

$$ct = \frac{\zeta}{2} (\sinh \theta - 1)$$

where the integration was again performed by substitution.

Note: θ here has *nothing* to do with the coordinate angle θ !

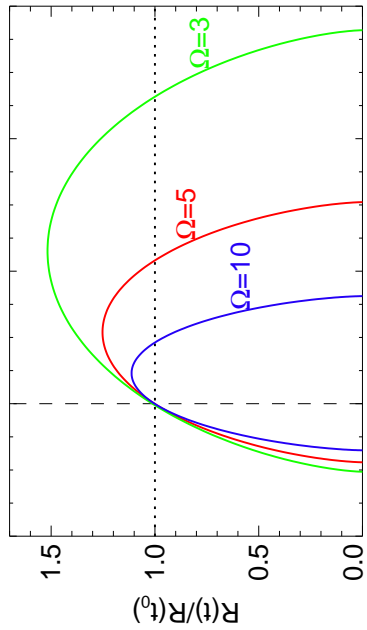
Dynamics

The cycloid solution shows that for $\Omega > 1$, the universe has a finite lifetime, i.e., it expands to a maximum and then becomes smaller and dies in a "big crunch". The maximum expansion occurs at $\theta = \pi$, with a maximum scale factor of

$$R_{\max} = \xi = \frac{c}{H_0} (\Omega_0 - 1)^{3/2} \quad (9.82)$$

The big crunch will happen at $\theta = 2\pi$, such that the lifetime of the closed universe is

$$t_{\text{life}} = \frac{\pi}{H_0} (\Omega_0 - 1)^{3/2} \quad (9.83)$$

 **$k = +1$, Matter dominated, I**

For the closed universe, one finds

$$R = \frac{\xi}{2} (1 - \cos \theta) \quad (9.78)$$

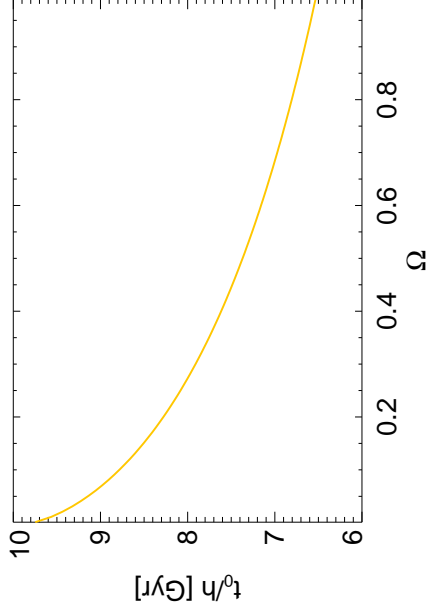
$$ct = \frac{\xi}{2} (\theta - \sin \theta)$$

Note that R is a cyclic function

\Rightarrow The closed universe has a finite lifetime, given by

$$t_{\text{life}} = \frac{\pi}{H_0} (\Omega_0 - 1)^{3/2} \quad (9.83)$$

Dynamics

 **$k = -1$, Matter dominated, II**

To obtain the age of the universe, note that at the present time,

$$\cosh \theta_0 = \frac{2 - \Omega_0}{\Omega_0}$$

$$\sinh \theta_0 = \frac{2}{\Omega_0} \sqrt{1 - \Omega_0} \quad (9.87)$$

(identical derivation as that leading to Eq. 9.79)

$$t_0 = \frac{1}{2H_0} \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} \cdot \left\{ \frac{2}{\Omega_0} \sqrt{1 - \Omega_0} - \ln \left(\frac{2 - \Omega_0 + 2\sqrt{1 - \Omega_0}}{\Omega_0} \right) \right\} \quad (9.88)$$

therefore,

Dynamics

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**Summary**

For the matter dominated case, our results from Eqs. (9.78), and (9.86) can be written with the functions S_k and C_k (Eq. 9.24) in form of the cycloid solution:

$$R = k\mathcal{R} (1 - C_k(\theta))$$

$$ct = k\mathcal{R} (\theta - S_k(\theta)) \quad (9.89)$$

with

$$S_k(\theta) = \begin{cases} \sin \theta & \text{for } k = +1 \\ \theta & \text{for } k = 0 \\ \sinh \theta & \text{for } k = -1 \end{cases} \quad \text{and} \quad C_k(\theta) = \begin{cases} \cos \theta & \text{for } k = +1 \\ 1 & \text{for } k = 0 \\ \cosh \theta & \text{for } k = -1 \end{cases} \quad (9.24)$$

and where the characteristic radius, \mathcal{R} , is given by

$$\mathcal{R} = \frac{c}{H_0} \frac{\Omega_0/2}{(k(\Omega_0 - 1))^{3/2}} \quad (9.90)$$

Notes:

- Eq. (9.89) can also be derived as the result of the Newtonian collapse/expansion of a spherical mass distribution.
- θ is called the **development angle**, it is equal to the **conformal time** (Eq. (9.32)).

Dynamics

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