# *Determination of $\Omega$ and $\Lambda$*



## Inflation

Previous lectures: Inflation requires

$$\Omega = \frac{\rho}{\rho_{\text{crit}}} = \Omega_{\text{m}} + \Omega_\Lambda = 1 \tag{12.1}$$

Here,

$\Omega_{\mathbf{m}}$**:** $\Omega$ due to gravitating stuff,

$\Omega_\Lambda$**:** $\Omega$ due to vacuum energy or other exotic stuff.

To decide whether that is true:

- need inventory of gravitating material in the universe,
- need to search for evidence of non-zero $\Lambda$

Also search for evidence in structure formation $\Longrightarrow$ Later…





## Inflation

Remember that

$$\Omega_{\text{m}} = \frac{\rho_{\text{m}}}{\rho_{\text{crit}}} = \frac{8\pi G\rho}{3H^2} \tag{8.42}$$

and

$$\Omega_\Lambda = \frac{\rho_{\text{vac}}}{\rho_{\text{crit}}} = \frac{\rho_{\text{vac}}}{3H^2/8\pi G} = \frac{\Lambda c^2}{3H^2} \tag{11.15}$$

As for a typical ensemble of stars,

$$\frac{M}{L} \approx \text{const.} \tag{12.2}$$

we often express $\Omega$ in terms of a mass to luminosity ratio:

Using canonical luminosity density of universe, one can show (Peacock, 1999, p. 368, for the B-band):

$$\left.\frac{M}{L}\right|_{\text{crit}} = 1390\, h \frac{M_\odot}{L_\odot} \tag{12.3}$$

…which means that there *must* be lots of dark matter.





## Introduction

Constituents of $\Omega_{\text{m}}$:

- Radiation (CMBR)
- Neutrinos
- Baryons ("normal matter", $\Omega_{\text{b}}$)
- Other, non-radiating, gravitating material ("dark matter")

*Radiation:* From temperature of CMBR, using $u = \rho c^2 = a_{\text{rad}} T^4$:

$$\Omega_\gamma h^2 = 2.480 \times 10^{-5} \tag{12.4}$$

for $h = 0.72$, $\Omega_\gamma = 4.8 \times 10^{-5}$

*Massless Neutrinos* have

$$\Omega_\nu = 3 \cdot \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \Omega_\gamma = 0.68\, \Omega_\gamma \tag{12.5}$$

Photons and massless neutrinos are unimportant for todays $\Omega$.

## Massive Neutrinos



Sudbury Neutrino Observator (SNO) and Super-Kamiokande: Neutrinos are not massless.

From neutrino decoupling and expansion:

**Current neutrino density: 113 neutrinos $\mathrm{cm}^{-3}$ per neutrino family.**

In terms of $\Omega$:

$$\Omega_\nu h^2 = \frac{\sum m_i}{93.2\,\mathrm{eV}} = \frac{M_\nu}{93.2\,\mathrm{eV}} \tag{12.6}$$

$\Longrightarrow$ For $h = 0.72$, $m \sim 16\,\mathrm{eV}$ would be sufficient to close universe

Current mass limits:

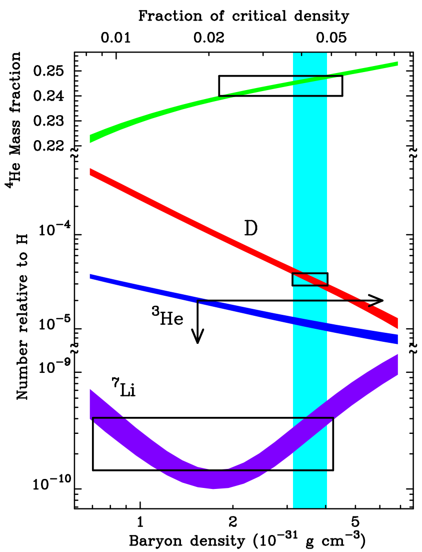
- cosmological studies (structure formation): $M_\nu < 0.17\,\mathrm{eV}$
- laboratory experiments: $m_{\nu_e} < 2.2\,\mathrm{eV}$, and solar neutrino oscillations imply $\Delta m = m_{\nu_\mu} - m_{\nu_e} \lesssim 10^{-4}\,\mathrm{eV}$.

See (Steidl, 2009) for a recent review on experiments to measure neutrino masses.



## Baryons



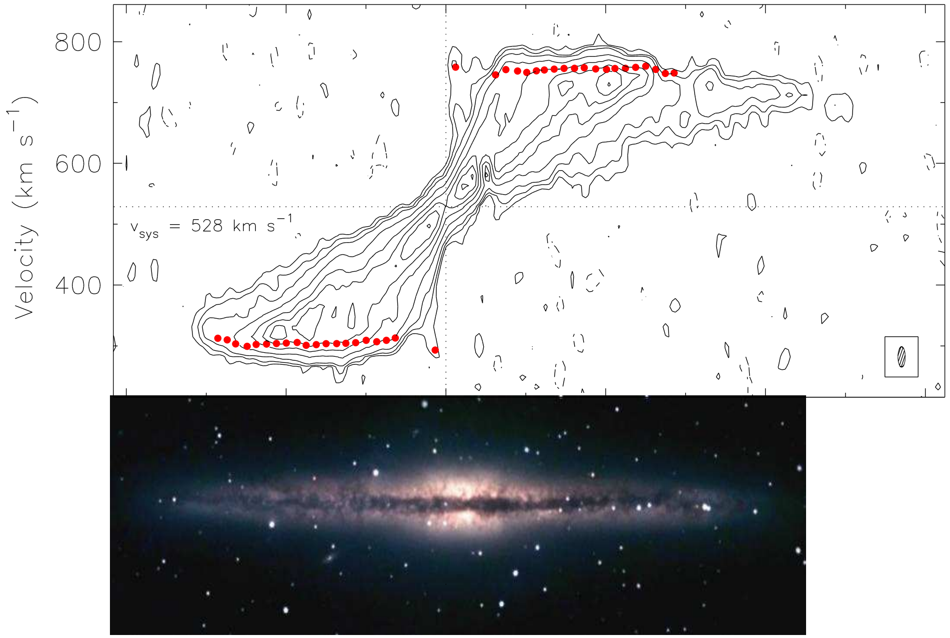



Best evidence for mass in baryons, $\Omega_b$: primordial nucleosynthesis.

$$\Omega_b h^2 = 0.02 \pm 0.002 \tag{12.7}$$

(Burles, Nollett & Turner, 1999, Fig. 1)






NGC 891 (Swaters et al., 1997, ApJ 491, 140 / Paul LeFevre, S&T Nov. 2002)

## Galaxy Rotation Curves, II

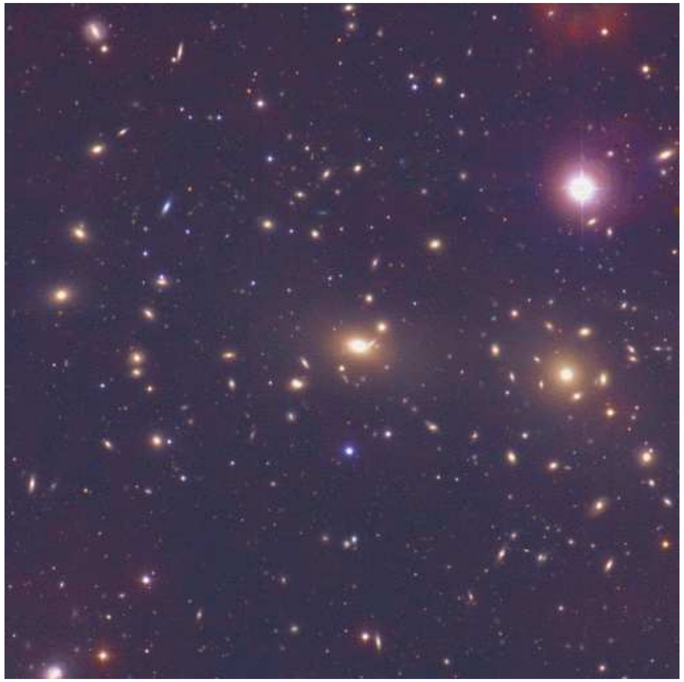


Reminder: galaxies have flat rotation curves. Analysis of mass required to describe motion leads to dark matter halos and to

$$\left.\frac{M}{L}\right|_{\mathrm{galaxies}} = 10\ldots 20\,\frac{M_\odot}{L_\odot} \tag{12.8}$$

**Only about 10% of the gravitating matter in universe radiates.**

NGC 891, KPNO 1.3 m
Barentine & Esquerdo



Coma Cluster (O. Lopez-Cruz, I. K. Shelton, & KPNO)



## Galaxy Clusters, II

To determine mass of galaxy clusters, use the virial theorem: In statistical equilibrium:

$$E_{\text{kin}} = -E_{\text{pot}}/2 \tag{12.9}$$

*Measurement:* assume isotropy, such that

$$\left\langle v^2 \right\rangle = \left\langle v_x^2 \right\rangle + \left\langle v_y^2 \right\rangle + \left\langle v_z^2 \right\rangle = 3\left\langle v_\parallel^2 \right\rangle \tag{12.10}$$

Assuming that the velocity dispersion is independent of $m_i$ gives:

$$E_{\text{kin}} = \frac{1}{2}\sum_i m_i \boldsymbol{v}_i^2 = \frac{3}{2}M\left\langle v_\parallel^2 \right\rangle \tag{12.11}$$

where $M$ is the total mass.

If the cluster is spherically symmetric $\Longrightarrow$ Define weighted mean separation $R_{\text{cl}}$, such that

$$E_{\text{pot}} = \frac{GM^2}{R_{\text{cl}}} \tag{12.12}$$

From Eqs. (12.11) and (12.12):

$$M = \frac{3}{G}\left\langle v_\parallel^2 \right\rangle R_{\text{cl}} \tag{12.13}$$

E.g.: $v_\parallel \sim 1000\,\text{km}\,\text{s}^{-1}$, $R \sim 1\,\text{Mpc} \Longrightarrow M = 1.4\times 10^{48}\,\text{g} = 7\times 10^{14}\,M_\odot$ (MW: $6\times 10^{11}\,M_\odot$).





### Derivation of the Virial Theorem

Assume system of particles, each with mass $m_i$. Acceleration on particle $i$:

$$\ddot{\mathbf{r}}_i = \sum_{j\neq i} \frac{Gm_j(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} \tag{12.14}$$

. . . scalar product with $m_i\mathbf{r}_i$

$$m_i\mathbf{r}_i \cdot \ddot{\mathbf{r}}_i = \sum_{j\neq i} \frac{Gm_im_j\mathbf{r}_i\cdot(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} \tag{12.15}$$

. . . since

$$\frac{1}{2}\frac{\mathrm{d}^2\mathbf{r}_i^2}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}_i\cdot\mathbf{r}_i) = \ddot{\mathbf{r}}_i\cdot\mathbf{r}_i + \dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i \tag{12.16}$$

. . . therefore Eq. (12.15)

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(m_i\mathbf{r}_i^2) - m_i\dot{\mathbf{r}}_i^{\,2} = \sum_{j\neq i} \frac{Gm_im_j\mathbf{r}_i\cdot(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} \tag{12.17}$$

Summing over all particles in the system gives

$$\frac{1}{2}\sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2}(m_i\mathbf{r}_i^2) - \sum_i m_i\dot{\mathbf{r}}_i^{\,2} = \sum_i\sum_{j\neq i} \frac{Gm_im_j\mathbf{r}_i\cdot(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} \tag{12.18}$$

$$= \frac{1}{2}\left(\sum_i\sum_{j\neq i} Gm_im_j\frac{\mathbf{r}_i\cdot(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3} + \sum_j\sum_{i\neq j} Gm_jm_i\frac{\mathbf{r}_j\cdot(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_j - \mathbf{r}_i|^3}\right) \tag{12.19}$$

$$= \frac{1}{2}\left(\sum_i\sum_{j\neq i} Gm_im_j\frac{\mathbf{r}_i\cdot\mathbf{r}_j - \mathbf{r}_i^2}{|\mathbf{r}_i - \mathbf{r}_j|^3} + \sum_j\sum_{i\neq j} Gm_jm_i\frac{\mathbf{r}_j\cdot\mathbf{r}_i - \mathbf{r}_j^2}{|\mathbf{r}_j - \mathbf{r}_i|^3}\right) \tag{12.20}$$

$$= -\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \frac{Gm_im_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{12.21}$$



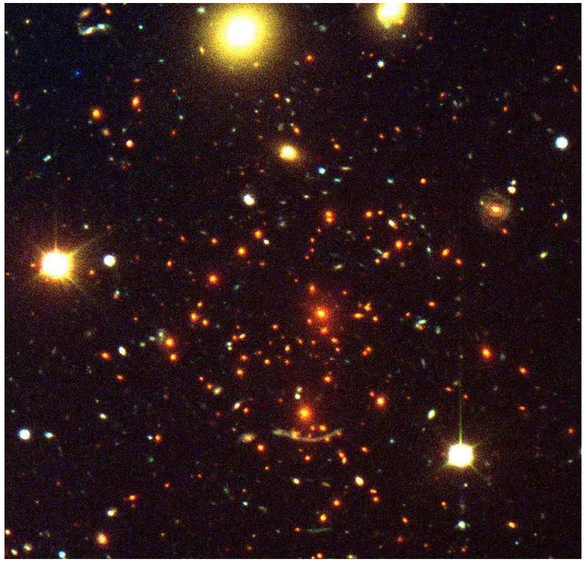
Thus, identifying the total kinetic energy, $T$, and the gravitational potential energy, $U$, gives

$$2T - U = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_i m_i\mathbf{r}_i^2 = 0 \tag{12.22}$$

in statistical equilibrium.

Thus we find the virial theorem: $T = \frac{1}{2}|U|$

# Galaxy Clusters, III



More detailed analysis using more complicated mass models gives (Merritt, 1987):

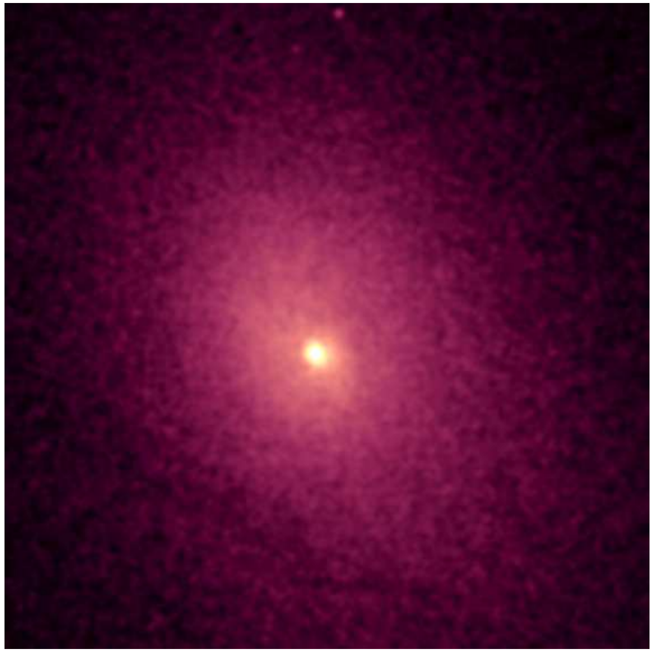
$$\frac{M}{L} \sim 350h^{-1}\frac{M_\odot}{L_\odot} \qquad (12.23)$$

while we would have expected $M/L = 10\dots20$ as for galaxies

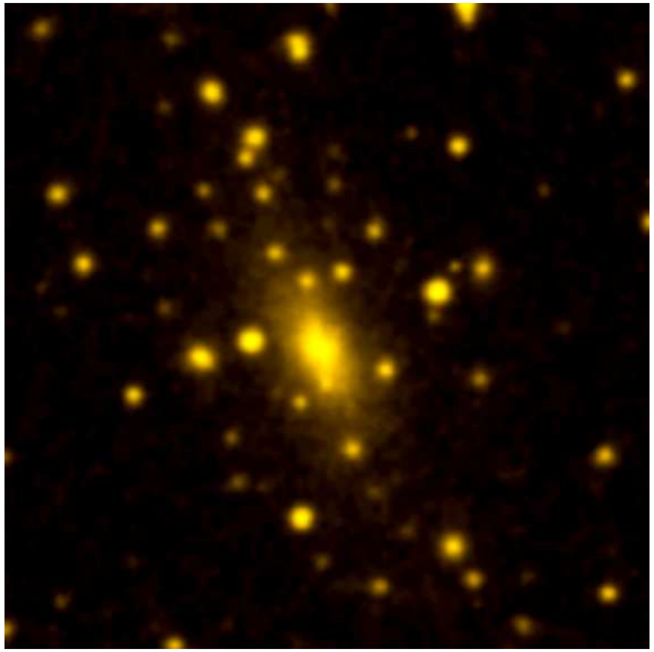
**Dark matter is an important constituent in galaxy clusters**

Abell 370 (VLT UT1+FORS)

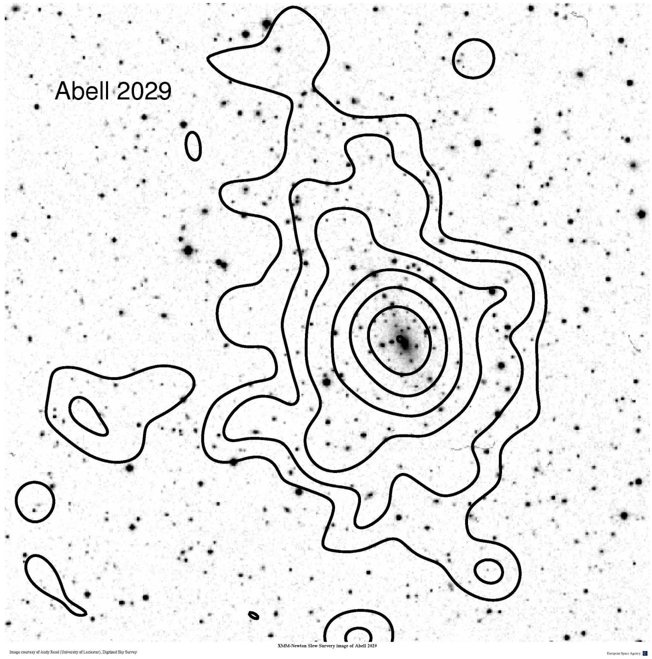




Abell 2029, Soft X-rays (*Chandra*; NASA/CXC/UCI/A.Lewis et al.)



Abell 2029, Palomar Schmidt [DSS]




Abell 2029, Optical and X-rays (*XMM-Newton*; Andy Read [Leicester]/DSS/ESA; larger FoV)

## X-ray emission, IV

X-ray emission from galaxy clusters gives mass to higher precision:

Assume gas in potential of galaxy cluster. If gas is in hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r\rho}{r^2} \tag{12.24}$$

where the pressure $P$ can be determined from the equation of state:

$$P = nkT = \frac{\rho kT}{\mu m_{\mathrm{H}}} \tag{12.25}$$

where $m_{\mathrm{H}}$: mass of H-atom, $\mu$ mean molecular weight of gas ($\mu = 0.6$ for fully ionized).

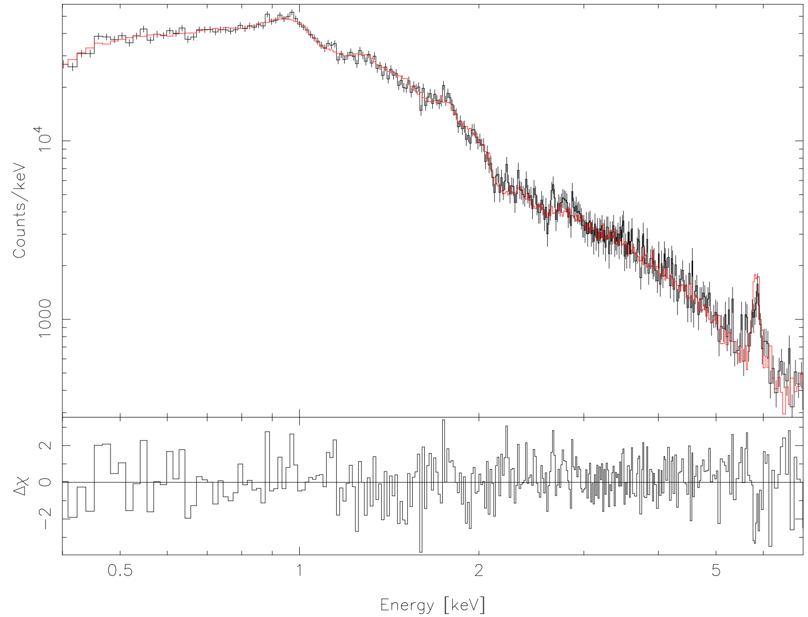
Differentiating Eq. (12.25) wrt $r$ gives

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{k}{\mu m_{\mathrm{H}}}\left(T\frac{\mathrm{d}\rho}{\mathrm{d}r} + \rho\frac{\mathrm{d}T}{\mathrm{d}r}\right) = \frac{\rho kT}{\mu m_{\mathrm{H}}}\left(\frac{\mathrm{d}\log\rho}{\mathrm{d}r} + \frac{\mathrm{d}\log T}{\mathrm{d}r}\right) \tag{12.26}$$

Inserting $\mathrm{d}P/\mathrm{d}r$ into Eq. (12.24) and solving for $M_r$ gives

$$M_r = -\frac{kTr^2}{G\mu m_{\mathrm{H}}}\left(\frac{\mathrm{d}\log\rho}{\mathrm{d}r} + \frac{\mathrm{d}\log T}{\mathrm{d}r}\right) \tag{12.27}$$





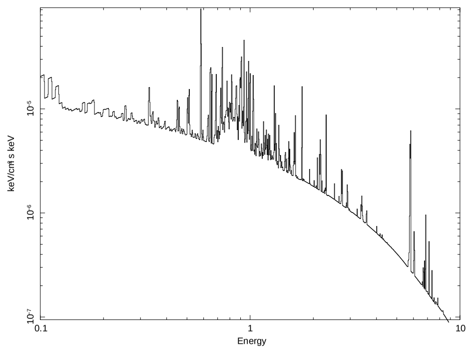

(Wise, McNamara & Murray, 2004, Fig. 2)

X-ray spectrum of A1068 obtained from *Chandra*



## X-ray emission, V

To determine $M_r$, we need to measure $T(r)$ and $\rho(r)$. These quantities can be obtained from the observed X-ray spectrum:




Theoretical X-ray spectrum of a cluster.

Cluster gas mainly radiates by bremsstrahlung emission, with a spectral continuum shape

$$\epsilon(E) \propto \left(\frac{m_{\mathrm{e}}}{kT}\right)^{1/2} g(E,T)\, n\, n_{\mathrm{e}} \exp\left(-\frac{E}{kT}\right) \tag{12.28}$$

where

$n$: number density of nuclei,

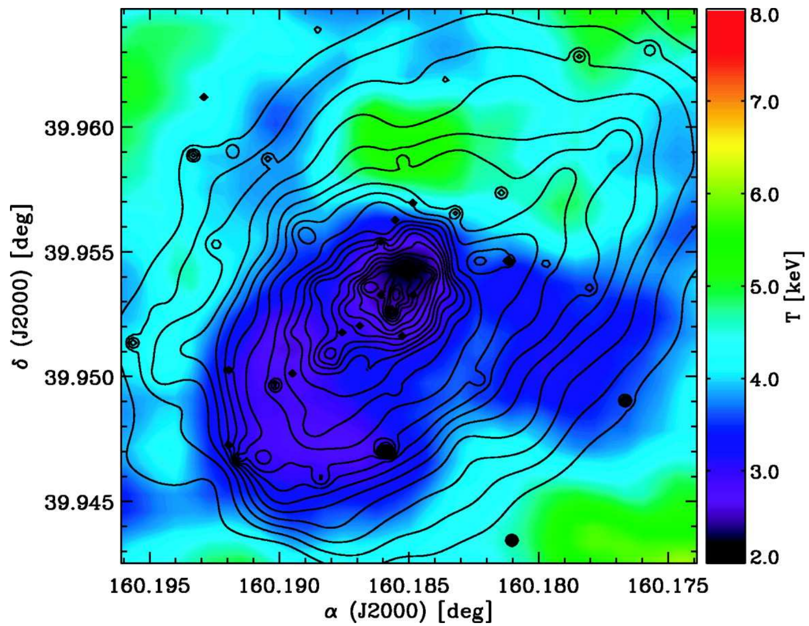
$n_{\mathrm{e}}$: number density of electrons,

$g(E,T)$: Gaunt factor (QM correction factor, roughly constant).

plus emission lines. . .

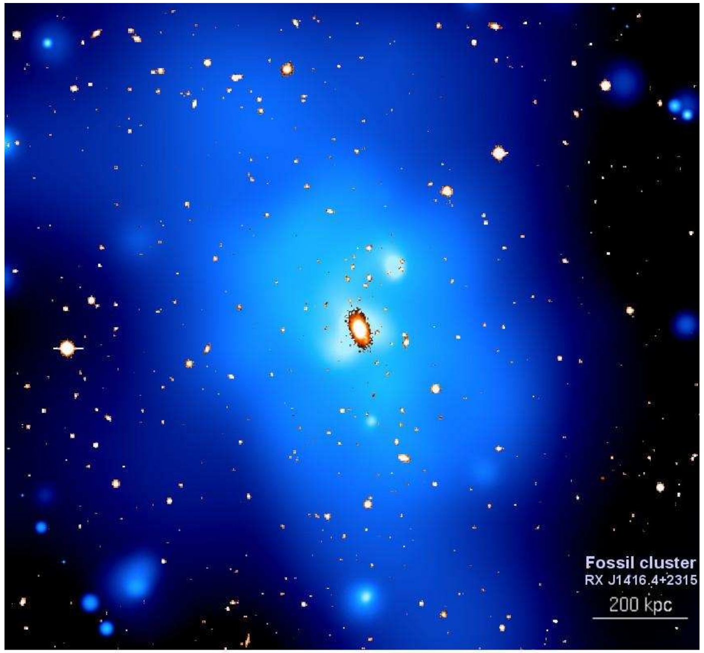
$\Longrightarrow$ $T(r)$ can be obtained from the X-ray spectral shape, $n$ and $n_{\mathrm{e}}$ from the measured flux $\Longrightarrow$ $M_r$.






(Wise, McNamara & Murray, 2004, Fig. 8)

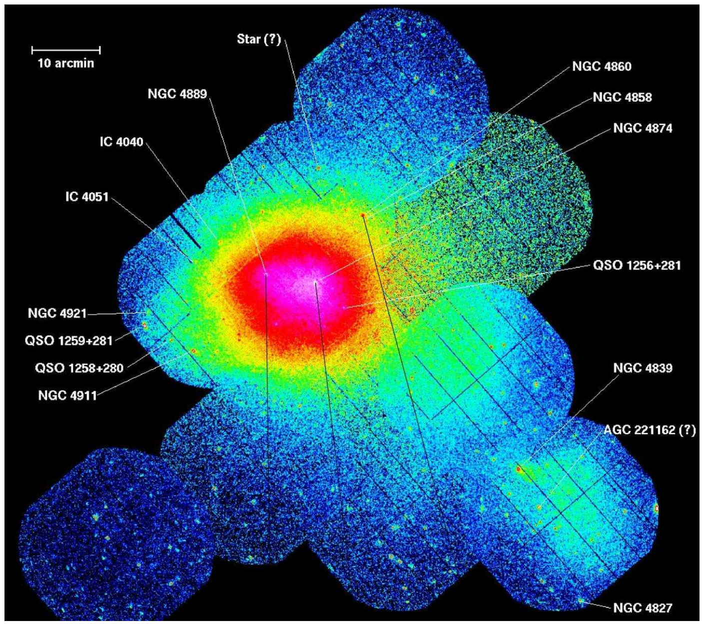
Temperature distribution in A1068 obtained with *Chandra*



XMM-Newton explores the fossil galaxy cluster RX J1416.4+2315

Image courtesy of Habib Khosroshahi (University of Birmingham)

European Space Agency




XMM-Newton, EPIC-pn

Technical problems:

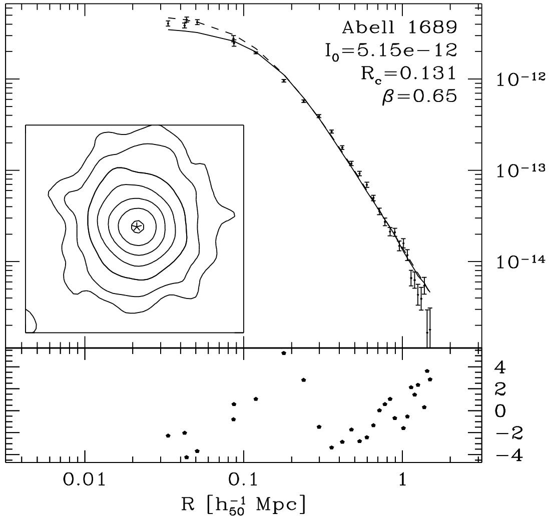
- see through cluster $\Longrightarrow$ integrate over line of sight, assuming spherical geometry.
- spherical geometry is assumed
- it is unclear whether gas is in hydrostatic equilibrium (cooling flows? – but note, there is sparse evidence for a "flow")

Result for Coma:

$$\frac{M_\mathrm{B}}{M_\mathrm{tot}} = 0.01 + 0.05\,h^{-3/2} \qquad (12.29)$$



## X-ray emission, X




(Mohr, Mathiesen & Evrard, 1999)

Generally: assume intensity profile from $\beta$-model,

$$\frac{I(r)}{I_0} = \left(1 + \left(\frac{r}{R_\mathrm{c}}\right)^2\right)^{-3\beta+\frac{1}{2}} \qquad (12.30)$$

and obtain $T$ from fitting X-ray spectra to "shells" $\Longrightarrow$ technically complicated...

Summary for X-ray mass determination for 45 clusters (Mohr, Mathiesen & Evrard, 1999):

$$f_\mathrm{gas} = (0.07 \pm 0.002)h^{-3/2} \qquad (12.31)$$

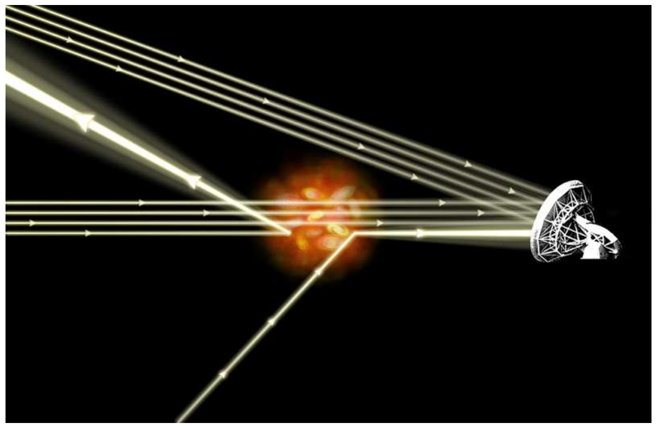
resulting in

$$\Omega_\mathrm{m} = \Omega_\mathrm{b}/f_\mathrm{gas} = (0.3 \pm 0.05)\,h^{-1/2} \qquad (12.32)$$





## Sunyaev-Zeldovich, I



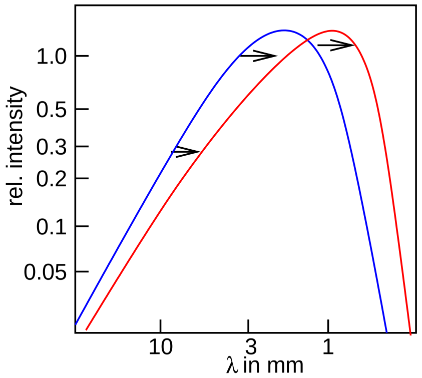
NASA/CXC/M.Weiss

Gas in cooling flow influences CMBR by Compton upscattering

$\Longrightarrow$ Sunyaev-Zeldovich effect (1970).

## Sunyaev-Zeldovich, II

The quantitative derivation of the SZ-effect is difficult, basically, one sets up the Fokker-Planck equation for the photon gas and from this derives the so-called Kompaneets equation, see, e.g., Peacock (1999, p. 375ff.).



after Schneider

The basic ingredients are the optical depth for Compton scattering (Compton $y$-parameter):

$$y = \int \left(\frac{kT_e}{m_e c^2}\right) \sigma_T N_e \, dl \qquad (12.33)$$

From this follows in the Rayleigh-Jeans regime that the intensity due to Compton upscattering changes as follows:

$$\frac{\Delta I}{I} = -2y \sim 10^{-4} \qquad (12.34)$$

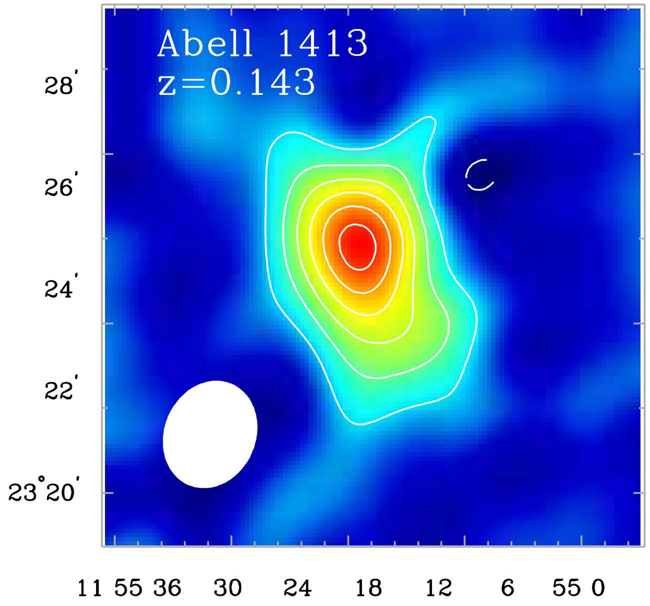
(for typical parameters).

$\Longrightarrow$ $\Delta I$ allows to measure of $\int N_e T_e \, dl$

$\Longrightarrow$ Mass!

$T$ is known from X-ray spectrum.





(temperature decrement from 3 K background, Carlstrom et al., 2000, Fig. 3)

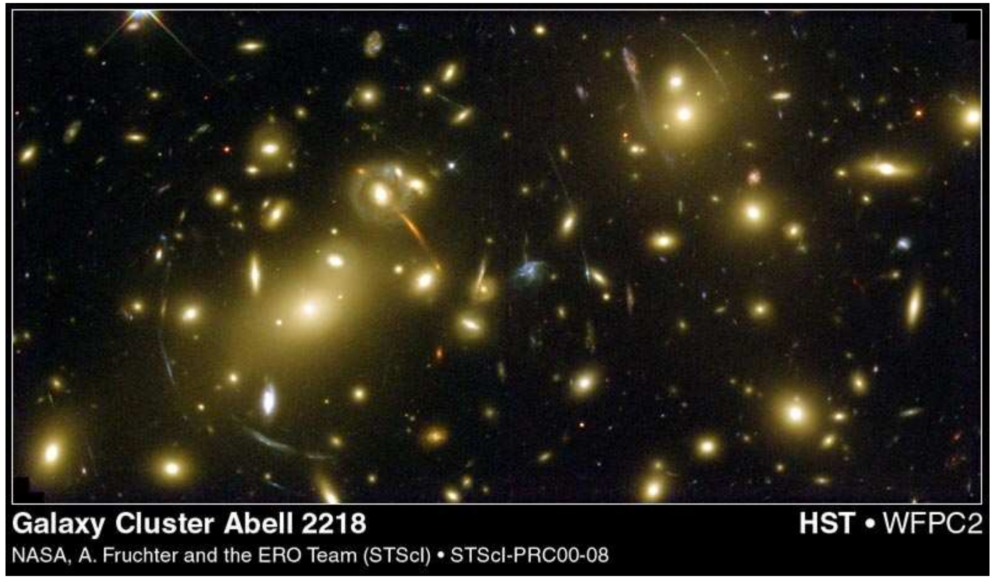
SZ analysis gives gas fraction for 27 clusters

$$f_{gas} = (0.06 \pm 0.006) h^{-3/2} \qquad (12.35)$$

remarkably similar to X-ray result $\Longrightarrow$ clumping of gas does not influence results! (SZ only traces real gas...)

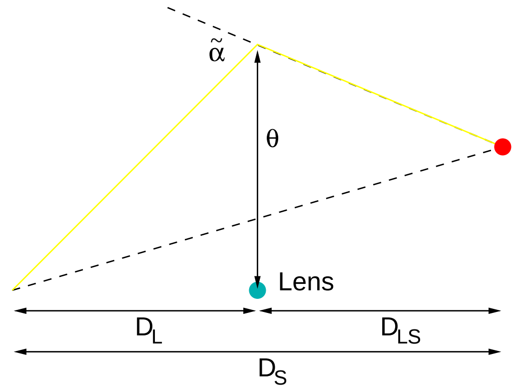
$f_{gas}$ translates to

$$\Omega_m = (0.25 \pm 0.04)\, h^{-1} \qquad (12.36)$$






## Gravitational Lenses, II



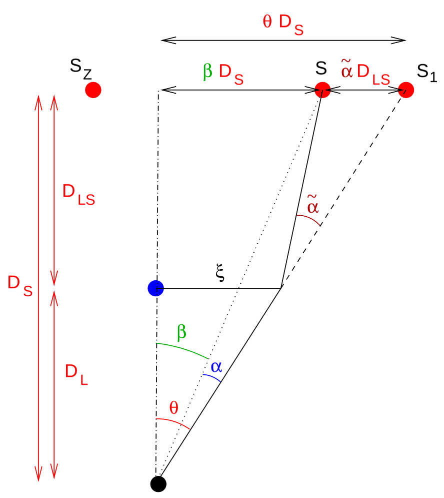
(after Longair, 1998, Fig. 4.8a)

GR: Angular deflection of light due to presence of mass $M$:

$$\tilde{\alpha} = \frac{4GM}{\theta c^2} = \frac{2}{c^2} \cdot \frac{2GM}{\theta} \qquad (12.37)$$

where $\theta$: distance of closest approach (*twice* the classical result).

## Gravitational Lenses, III




after Wambsganss (1998, Fig. 3)

In the small angle approximation:

$$\theta D_{\rm S} = \beta D_{\rm S} + \tilde{\alpha} D_{\rm LS} \quad (12.38)$$

such that

$$\beta = \theta - \frac{D_{\rm LS}}{D_{\rm S}}\tilde{\alpha} \quad (12.39)$$

Defining the reduced deflection angle,

$$\alpha = \frac{D_{\rm LS}}{D_{\rm S}}\tilde{\alpha} = \frac{D_{\rm LS}}{D_{\rm S}} \cdot \frac{2}{c^2} \cdot \frac{2GM}{\theta} \quad (12.40)$$

then gives the lens equation

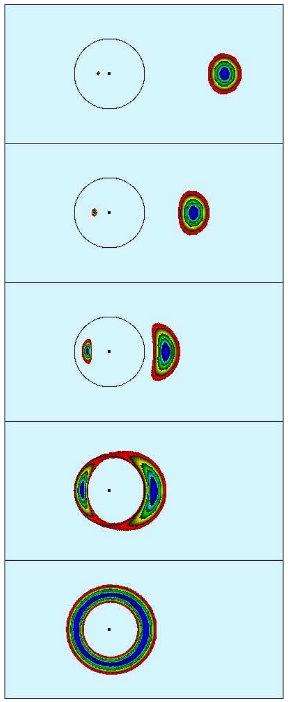
$$\beta = \theta - \alpha = \theta - \frac{D_{\rm LS}}{D_{\rm L} D_{\rm S}} \cdot \frac{4GM}{c^2\theta} = \theta - \frac{1}{D} \cdot \frac{4GM}{c^2\theta} \quad (12.41)$$

where

$$D = \frac{D_{\rm L} D_{\rm S}}{D_{\rm LS}} \quad (12.42)$$

(last expression valid for a point-mass)





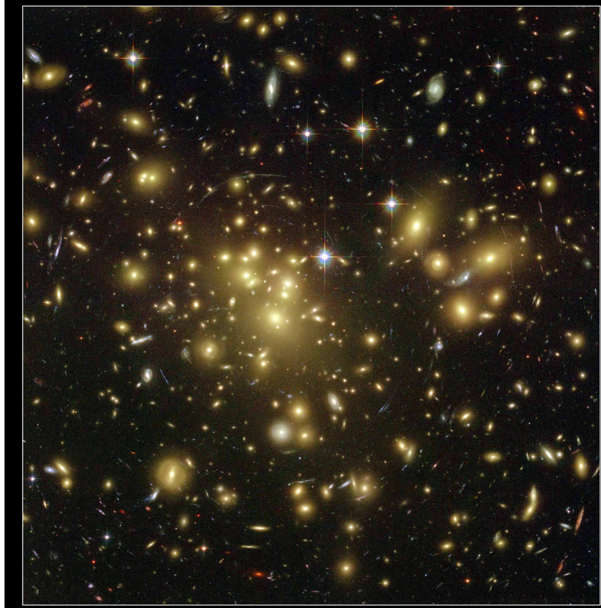
Einstein ring: source directly behind lens,
Obtain radius by setting $\beta = 0$ in lens-equation (Eq. 12.41):

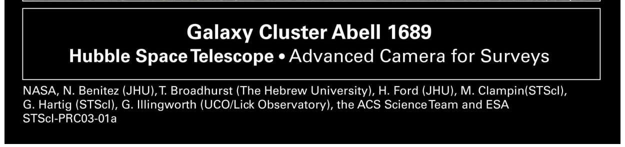
$$\theta_{\rm E}^2 = \frac{4GM}{c^2}\frac{1}{D} \quad (12.43)$$

i.e.,

$$\theta_{\rm E} = 98.9'' \left(\frac{M}{10^{15}\,M_\odot}\right)^{1/2} \frac{1}{(D/1\,{\rm Gpc})^{1/2}} \quad (12.44)$$

Mass measurements possible by observing "giant luminous arcs" and Einstein rings.






General results of mass determinations from lensing agree with other methods.



## Summary

So far, we have seen:

**Photons:**

$$\Omega_\gamma h^2 = 2.480 \times 10^{-5} \quad (12.45)$$

**Neutrinos:**

$$\Omega_\nu h^2 = 1.69 \times 10^{-5} \quad (12.46)$$

**Baryons** (from nucleosynthesis):

$$\Omega_{\rm b} h^2 = 0.02 \quad \text{where} \quad \Omega_{\rm stars} \sim 0.005 \ldots 0.01 \quad (12.47)$$

**Baryons+dark matter** (from clusters):

$$\Omega_{\rm m} \sim 0.25 \quad (12.48)$$

(of which $\sim 10\%$ in baryons)

If we believe in $\Omega_{\rm total} \equiv 1 \Longrightarrow \Omega_\Lambda \sim 0.7$.

## Introduction

Clusters and galaxies: $\Omega_\mathrm{m} \sim 0.3$, but for baryons $\Omega_\mathrm{b} \sim 0.02$

$\Longrightarrow$ Rest of gravitating material is dark matter.

$\Longrightarrow$ There are two dark matter problems:

$$\Omega_\mathrm{m} \xleftarrow{\text{nonbaryonic dark matter}} \Omega_\mathrm{b} \xleftarrow{\text{baryonic dark matter}} \Omega_\mathrm{stars}$$

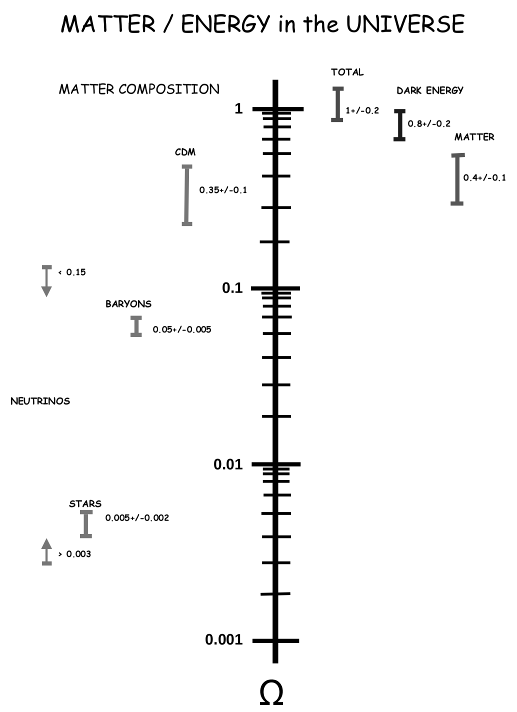
**baryonic dark matter=** undetected baryons:

- diffuse hot gas?
- MACHOs (Massive compact halo objects; white dwarfs, neutron stars, black holes, brown dwarfs, jupiters,. . . )

**nonbaryonic dark matter=** exotic stuff:

- massive neutrinos
- axions
- neutralinos






(Turner, 1999, Fig. 1, numbers slightly different to ours. . . )



## Baryonic Dark Matter, I

Intra Cluster Gas:

**Pro:**

1. same location where the hot gas in clusters also found,
2. structure formation suggests most baryons are *not* in structures today

**Contra:**

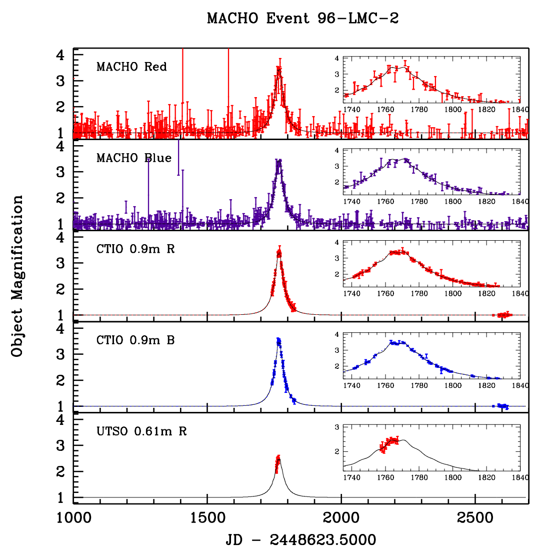
1. 90% of the universe is *not* in clusters. . .
2. gas has not been detected at any wavelength

If gas cold enough, would not expect it to be detectable, so point 2 is not really valid.





## Baryonic Dark Matter, II




MACHOS:

**Pro:**

1. detected by microlensing towards SMC and LMC (see figure) $\Longrightarrow$ MW halo consists of 50% WD

**Contra:**

1. possible "self-lensing" (by stars in MW or SMC/LMC; confirmed for a few cases)
2. where are white dwarfs?
3. WD formation rate too high (100 year$^{-1}$ Mpc$^{-3}$)

(Alcock et al., 2001, Fig. 2)

## Nonbaryonic Dark Matter

Nonbaryonic dark matter:

*Requirements:* must be gravitating and non-interacting with baryons

$\Longrightarrow$ Grab-box of elementary particle physics:

1. Neutrinos with non-zero mass
   **Pro:** It exists, mass limits are a few eV, need only $\langle m_\nu \rangle \sim 10\,\mathrm{eV}$
   **Contra:** $\nu$ are relativistic $\Longrightarrow$ Hot dark matter $\Longrightarrow$ Forces top down structure formation, contrary to what is believed to have happened.
2. Axion
   (=Goldstone boson from QCD, invented to prevent strong CP violation in QCD; $m \sim 10^{-5\ldots-2}\,\mathrm{eV}$)
   **Pro:** It *could* exist, would be in Bose-Einstein condensate due to inflation ($\Longrightarrow$ Cold dark matter!), might be detectable in the next 10 years
   **Contra:** We do not know it exists. . .
3. Neutralino or other WIMPs (weakly interacting massive particles; masses $m \sim \mathrm{GeV}$)
   **Pro:** Also is CDM
   **Contra:** We do not know they exist. . .





## Friedmann with $\Lambda \neq 0$, I

Since $\Omega_\mathrm{m} \sim 0.3$, if $\Omega_\mathrm{tot} = 1$, then $\Omega_\Lambda \sim 0.7 \Longrightarrow$ We need to study cosmology with $\Lambda \neq 0$.

*Reviews:* Carroll, Press & Turner (1992), Carroll (2001)

Friedmann equation with $\Lambda \neq 0$:

$$H^2(t) = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \quad (11.31)$$

And define the $\Omega$'s (Eqs. 8.42, 11.15):

$$\Omega_\mathrm{m} = \frac{8\pi G\rho_\mathrm{m}}{3H_0^2}, \qquad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}, \qquad \Omega_k = -\frac{kc^2}{R_0^2 H_0^2} \quad (12.49)$$

Because of Eq. (11.31),

$$\Omega_\mathrm{m} + \Omega_\Lambda + \Omega_k = \Omega + \Omega_k = 1 \quad (12.50)$$





## Friedmann with $\Lambda \neq 0$, II

It is easier to set $c = 1$ and to work with the dimensionless scale factor,

$$a = \frac{R(t)}{R_0} \quad (8.29)$$

$\Longrightarrow$ Friedmann:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_{\mathrm{m},0}}{a^3} - \frac{k}{a^2 R_0^2} + \frac{\Lambda}{3} \quad (12.51)$$

since $\rho_\mathrm{m} = \rho_{\mathrm{m},0}a^{-3}$ (Eq. 8.63).

Inserting the $\Omega$'s

$$\left(\frac{\dot{a}/H_0}{a}\right)^2 = \frac{\Omega_\mathrm{m}}{a^3} + \frac{1 - \Omega_\mathrm{m} - \Omega_\Lambda}{a^2} + \Omega_\Lambda \quad (12.52)$$

Substituting the time in units of todays Hubble time,

$$\tau = H_0 \cdot t \quad (12.53)$$

results in

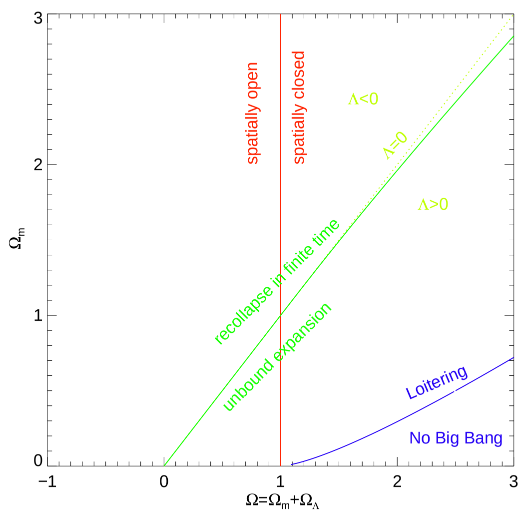
$$\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 = 1 + \Omega_\mathrm{m}\left(\frac{1}{a} - 1\right) + \Omega_\Lambda(a^2 - 1) \quad \text{where} \quad a(\tau = 1) = 1 \quad \text{and} \quad \left.\frac{\mathrm{d}a}{\mathrm{d}\tau}\right|_{\tau=1} = 1 \quad (12.54)$$

For most combinations of $\Omega_\mathrm{m}$ and $\Omega_\Lambda$, we need to solve the equations numerically.





## Friedmann with $\Lambda \neq 0$, III

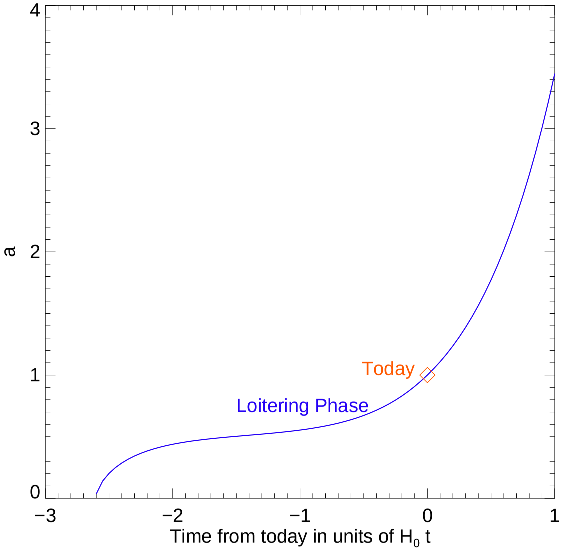



(after Carroll, Press & Turner, 1992, Fig. 1)

With $\Lambda$, evolution of universe is more complicated than without:

- unbound expansion possible for $\Omega < 1$,
- For $\Omega_\Lambda$ large: no big bang!
- For $\Omega_\Lambda$ large: possible "loitering phase"

## $\Omega_\Lambda > 1$, I




For large $\Omega_\Lambda$: contraction from $+\infty$ and reexpansion
$\Longrightarrow$ no big bang.
For slightly smaller $\Omega_\Lambda$: phase where $\dot{a} \sim 0$ in the past
$\Longrightarrow$ loitering universe.

"Loitering universe" with $\Omega_m = 0.55$, $\Omega_\Lambda = 2.055$





## $\Omega_\Lambda > 1$, II

QSO at $z = 5.82$, courtesy SDSS

Threshold for presence of a turning-point (Carroll, Press & Turner, 1992, Eq. 12):

$$\Omega_\Lambda \geq \Omega_{\Lambda,\text{thresh}} = 4\Omega_m \left\{ C_\kappa \left[ \frac{1}{3} C_\kappa^{-1} \left( \frac{1-\Omega_m}{\Omega_m} \right) \right] \right\}^3 \quad (12.55)$$

where $\kappa = \text{sgn}(0.5 - \Omega_m)$ and $C_\kappa(\theta)$ was defined in Eq. (8.24).
For $\Omega_\Lambda = \Omega_{\Lambda,\text{thresh}}$: turning-point, i.e., there is a minimal $a$.

Since $1 + z = 1/a$ (Eq. 8.53), existence of turning-point $\Longrightarrow$ maximal possible $z$:

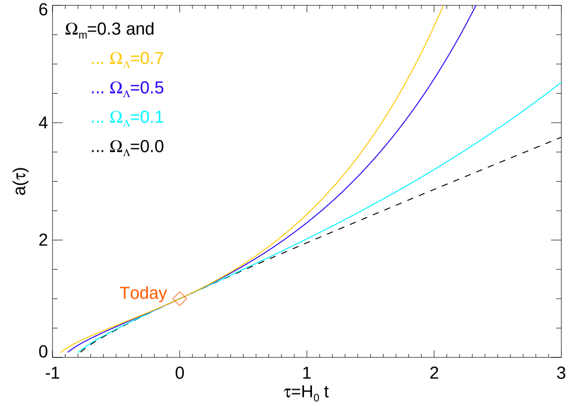
$$z \leq 2C_\kappa \left( \frac{1}{3} C_\kappa^{-1} \left\{ \frac{1-\Omega_m}{\Omega_m} \right\} \right) - 1 \quad (12.56)$$

(Carroll, Press & Turner, 1992, Eq. 14). Since quasars are observed with $z > 5.82$, this means that $\Omega_m < 0.007$, clearly not what is observed $\Longrightarrow \Omega_\Lambda < 1$.





## $\Omega_\Lambda < 1$




For $\Omega_\Lambda < 1$ evolution has two parts:

- matter domination, similar to earlier results
- $\Lambda$ domination, exponential rise.

Exponential rise called by some workers the "second inflationary phase"...





## $\Omega_\Lambda < 1$

Calculation of age of universe is similar to $\Omega_\Lambda = 0$ case (see, e.g., Eq. 8.81), but generally only possible numerically.

*Result:*

**Universes with $\Omega_\Lambda > 0$ are *older* than those with $\Omega_\Lambda = 0$.**

This solves the age problem, that some globular clusters have age comparable to age of universe if $\Omega_\Lambda = 0$.

Analytical formula for age (Carroll, Press & Turner, 1992, Eq. 17):

$$t = \frac{2}{3H_0} \frac{\sinh^{-1}\left(\sqrt{(1-\Omega_a)/\Omega_a}\right)}{\sqrt{1-\Omega_a}} \quad (12.57)$$

for $\Omega_a < 1$, where

$$\Omega_a = 0.7\Omega_m + 0.3(1 - \Omega_\Lambda) \quad (12.58)$$

For $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$: $t = 13.5\,\text{Gyr}$.

Remember that for $\Omega_m = 1$, $t = 3/2H_0$!

## Luminosity Distance, I

Influence of $\Lambda$ is most prominent at large distances!

$\Longrightarrow$ Expect influence on Hubble Diagram.

$\Longrightarrow$ Need to find relation between measured flux, emitted luminosity, and redshift.

Assume source with luminosity $L$ at comoving coordinate $r$, emitting isotropically into $4\pi$ sr.

At time of detection today, photons are

- on sphere with proper radius $R_0 r$,
- redshifted by factor $1+z$,
- spread in time by factor $1+z$.

$\Longrightarrow$ observed flux is

$$F = \frac{L}{4\pi R_0^2 r^2 (1+z)^2} \qquad (12.59)$$





## Luminosity Distance, II

Because the observed flux is

$$F = \frac{L}{4\pi R_0^2 r^2 (1+z)^2} \qquad (12.59)$$

in analogy to the inverse square law one defines the luminosity distance as

$$d_\mathrm{L} = R_0 \cdot r \cdot (1+z) \qquad (12.60)$$

The calculation of $d_\mathrm{L}$ is somewhat technical, one can show that (Carroll, Press & Turner, 1992):

$$d_\mathrm{L} = \frac{c}{H_0}|\Omega_k|^{-1/2} \cdot S_{-\mathrm{sgn}(\Omega_k)}\left\{|\Omega_k|^{1/2}\int_0^z \left[(1+z)^2(1+\Omega_\mathrm{m}z) - z(2+z)\Omega_\Lambda\right]^{1/2} \,\mathrm{d}z\right\} \qquad (12.61)$$





## Supernovae

Best way to determine $\Omega_\Lambda$:

Type Ia supernovae

Remember: SN Ia = CO WD collapse… (Hoyle, Fowler, Colgate, Wheeler,…)

The distance modulus is

$$m - M = 5\log\left(\frac{d_\mathrm{L}}{1\,\mathrm{Mpc}}\right) + 25 \qquad (12.62)$$

Use SNe as standard candles $\Longrightarrow$ Deviations from $d_\mathrm{L} \propto z$ indicative of $\Lambda$.

Two projects:

- High-$z$ Supernova Team (STSCI, Riess et al.)
- Supernova Cosmology Project (LBNL, Perlmutter et al.)

Both find SNe out to $z \sim 1$.

Present mainly Perlmutter et al. results here, Riess et al. (1998) are similar.