World Models

Structure

Observations: cosmological principle holds: The universe is homogeneous and isotropic.

$\Longrightarrow$ Need theoretical framework obeying the cosmological principle.

Use combination of

- General Relativity
- Thermodynamics
- Quantum Mechanics

$\Longrightarrow$ Complicated!

For 99% of the work, the above points can be dealt with separately:

1. Define metric obeying cosmological principle.
2. Obtain equation for evolution of universe using Einstein field equations.
3. Use thermo/QM to obtain equation of state.
4. Solve equations.

GRT vs. Newton

Before we can start to think about universe: Brief introduction to assumptions of general relativity.

$\Longrightarrow$ See theory lectures for the gory details, or check with the literature (Weinberg or MTW).

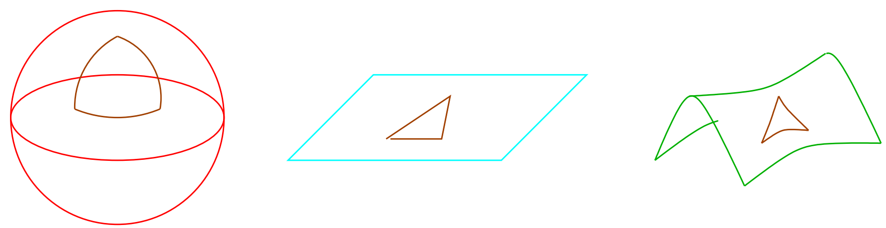
Assumptions of GRT:

- Space is 4-dimensional, might be curved
- Matter (=Energy) modifies space (Einstein field equation).
- Covariance: physical laws must be formulated in a coordinate-system independent way.
- Strong equivalence principle: There is no experiment by which one can distinguish between free falling coordinate systems and inertial systems.
- At each point, space is locally Minkowski (i.e., locally, SRT holds).

$\Longrightarrow$ Understanding of geometry of space necessary to understand physics.

2D Metrics

Before describing the 4D geometry of the universe: first look at 2D spaces (easier to visualize).



After Silk (1997, p. 107)

There are three classes of isotropic and homogeneous two-dimensional spaces:

- 2-sphere ($\mathscr{S}^2$) positively curved
- x-y-plane ($\mathbb{R}^2$) zero curvature
- hyperbolic plane ($\mathscr{H}^2$) negatively curved

(curvature $\approx \sum$angles in triangle $>$, $=$, or $< 180^\circ$)

We will now calculate what the metric for these spaces looks like.

2D Metrics

The metric describes the local geometry of a space.

Differential distance, $\mathrm{d}s$, in Euclidean space, $\mathbb{R}^2$:

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 \quad (9.1)$$

The metric tensor, $g_{\mu\nu}$, is defined through

$$\mathrm{d}s^2 = \sum_\mu \sum_\nu g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu =: g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \quad (9.2)$$

(Einstein's summation convention)

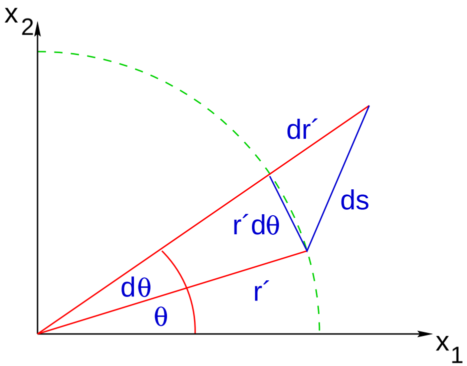
Thus, for the $\mathbb{R}^2$,

$$g_{11} = 1 \qquad g_{12} = 0$$
$$g_{21} = 0 \qquad g_{22} = 1 \quad (9.3)$$

2D Metrics

But: Other coordinate-systems are also possible in the plane!

Changing to polar coordinates r', θ, defined by



$$x_1 =: r'\cos\theta$$
$$x_2 =: r'\sin\theta \quad (9.4)$$

it is easy to see that

$$\mathrm{d}s^2 = \mathrm{d}r'^2 + r'^2\, \mathrm{d}\theta^2 \quad (9.5)$$

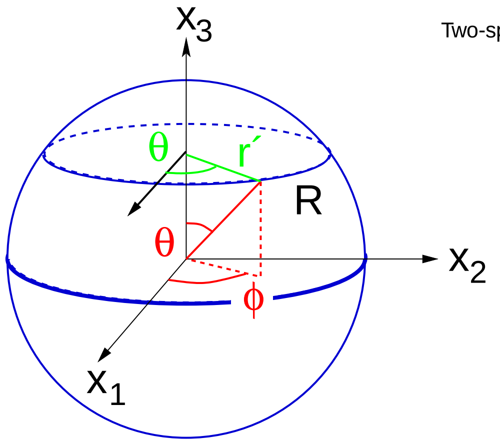
Performing a change of scale by substituting $r' = Rr$, then gives

$$\mathrm{d}s^2 = R^2\{\mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2\} \quad (9.6)$$

2D Metrics

A more complicated case occurs if space is curved.

Easiest case: surface of three-dimensional sphere (a two-sphere).



After Kolb & Turner (1990, Fig. 2.1)

Two-sphere with radius R in $\mathbb{R}^3$:

$$x_1^2 + x_2^2 + x_3^2 = R^2 \quad (9.7)$$

Length element of $\mathbb{R}^3$:

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$$

Eq. (9.7) gives

$$x_3 = \sqrt{R^2 - x_1^2 - x_2^2}$$

such that

$$\mathrm{d}x_3 = \frac{\partial x_3}{\partial x_1}\, \mathrm{d}x_1 + \frac{\partial x_3}{\partial x_2}\, \mathrm{d}x_2 = -\frac{x_1\, \mathrm{d}x_1 + x_2\, \mathrm{d}x_2}{\sqrt{R^2 - x_1^2 - x_2^2}} \quad (9.8)$$

2D Metrics

Introduce again polar coordinates r', θ in x_3-plane:

$$x_1 =: r'\cos\theta \quad x_2 =: r'\sin\theta \quad (9.4)$$

(note: r', θ are only unique in upper or lower half-sphere)

The differentials are given by

$$\mathrm{d}x_1 = \cos\theta\, \mathrm{d}r' - r'\sin\theta\, \mathrm{d}\theta \quad \text{and} \quad \mathrm{d}x_2 = \sin\theta\, \mathrm{d}r' + r'\cos\theta\, \mathrm{d}\theta \quad (9.9)$$

In cartesian coordinates, the length element on $\mathscr{S}^2$ is

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \frac{(x_1\, \mathrm{d}x_1 + x_2\, \mathrm{d}x_2)^2}{R^2 - x_1^2 - x_2^2} \quad (9.10)$$

inserting eq. (9.9) gives after some algebra

$$= r'^2\, \mathrm{d}\theta^2 + \frac{R^2}{R^2 - r'^2}\, \mathrm{d}r'^2 \quad (9.11)$$

finally, defining $r = r'/R$ (i.e., $0 \le r \le 1$) results in

$$\mathrm{d}s^2 = R^2\left\{\frac{\mathrm{d}r^2}{1 - r^2} + r^2\, \mathrm{d}\theta^2\right\} \quad (9.12)$$

2D Metrics

Alternatively, we can work in spherical coordinates on $\mathscr{S}^2$

$$\begin{aligned} x_1 &= R\sin\theta\cos\phi \\ x_2 &= R\sin\theta\sin\phi \\ x_3 &= R\cos\theta \end{aligned} \tag{9.13}$$

($\theta \in [0,\pi]$, $\phi \in [0, 2\pi]$).

Going through the same steps as before, we obtain after some tedious algebra

$$\mathrm{d}s^2 = R^2\left\{\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right\} \tag{9.14}$$

2D Metrics

(Important) remarks:

1. The 2-sphere has no edges, has no boundaries, but has still a finite volume, $V = 4\pi R^2$.
2. Expansion or contraction of sphere caused by variation of $R \Longrightarrow R$ determines the *scale* of volumes and distances on $\mathscr{S}^2$.

   R is called the *scale factor*

3. Positions on $\mathscr{S}^2$ are defined, e.g., by r and θ, *independent* on the value of R

   r and θ are called *comoving coordinates*

4. Although the metrics Eq. (9.10), (9.12), and (9.14) look *very* different, they still describe the *same* space $\Longrightarrow$ that's why physics should be covariant, i.e., independent of the coordinate system!

2D Metrics

The hyperbolic plane, $\mathscr{H}^2$, is defined by

$$x_1^2 + x_2^2 - x_3^2 = -R^2 \tag{9.15}$$

If we work in Minkowski space, where

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2 \tag{9.16}$$

then

$$= \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \frac{(x_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}x_2)^2}{R^2 + x_1^2 + x_2^2} \tag{9.17}$$

$\Longrightarrow$ substitute $R \rightarrow iR$ (where $i = \sqrt{-1}$) to obtain same form as for sphere (eq. 9.11)!

Therefore,

$$\mathrm{d}s^2 = R^2\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2\,\mathrm{d}\theta^2\right\} \tag{9.18}$$

2D Metrics

The analogy to spherical coordinates on the hyperbolic plane are given by

$$\begin{aligned} x_1 &= R\sinh\theta\cos\phi \\ x_2 &= R\sinh\theta\sin\phi \\ x_3 &= R\cosh\theta \end{aligned} \tag{9.19}$$

($\theta \in [-\infty, +\infty]$, $\phi \in [0, 2\pi]$).

A session with Maple (see handout) will convince you that these coordinates give

$$\mathrm{d}s^2 = R^2\left\{\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2\right\} \tag{9.20}$$

Remark:

$\mathscr{H}^2$ is unbound and has an infinite volume.

Transcript of Maple session to obtain Eq. (9.20):

```
> x1:=r*sinh(theta)*cos(phi);
```
$$x1 := r\sinh(\theta)\cos(\phi)$$
```
> x2:=r*sinh(theta)*sin(phi);
```
$$x2 := r\sinh(\theta)\sin(\phi)$$
```
> x3:=r*cosh(theta);
```
$$x3 := r\cosh(\theta)$$
```
> dx1:=diff(x1,theta)*dtheta+diff(x1,phi)*dphi;
```
$$dx1 := r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi$$
```
> dx2:=diff(x2,theta)*dtheta+diff(x2,phi)*dphi;
```
$$dx2 := r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi$$
```
> ds2:=dx1*dx1+dx2*dx2-(x1*dx1+x2*dx2)^2/(r^2+x 1^2+x2^2);
```
$$ds2 := (r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi)^2 + (r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi)^2 - (r\sinh(\theta)\cos(\phi)(r\cosh(\theta)\cos(\phi)\,dtheta - r\sinh(\theta)\sin(\phi)\,dphi) + r\sinh(\theta)\sin(\phi)(r\cosh(\theta)\sin(\phi)\,dtheta + r\sinh(\theta)\cos(\phi)\,dphi))^2 \Big/ (r^2 + r^2\sinh(\theta)^2\cos(\phi)^2 + r^2\sinh(\theta)^2\sin(\phi)^2)$$
```
> expand(ds2);
```
$$r^2\cosh(\theta)^2\cos(\phi)^2\,dtheta^2 + r^2\sinh(\theta)^2\sin(\phi)^2\,dphi^2 + r^2\cosh(\theta)^2\sin(\phi)^2\,dtheta^2 + r^2\sinh(\theta)^2\cos(\phi)^2\,dphi^2 - \frac{r^4\sinh(\theta)^2\cos(\phi)^4\cosh(\theta)^2\,dtheta^2}{\%1} - 2\,\frac{r^4\sinh(\theta)^2\cos(\phi)^2\cosh(\theta)^2\,dtheta^2\sin(\phi)^2}{\%1} - \frac{r^4\sinh(\theta)^2\sin(\phi)^4\cosh(\theta)^2\,dtheta^2}{\%1}$$
$$\%1 := r^2 + r^2\sinh(\theta)^2\cos(\phi)^2 + r^2\sinh(\theta)^2\sin(\phi)^2$$
```
> simplify(",{cosh(theta)^2-sinh(theta)^2=1}, [cosh(theta)]);
```
$$r^2\,dtheta^2 + r^2\sinh(\theta)^2\,dphi^2$$

2D Metrics

To summarize:

Sphere: $$\mathrm{d}s^2 = R^2\left\{\frac{\mathrm{d}r^2}{1-r^2} + r^2\,\mathrm{d}\theta^2\right\} \quad (9.12)$$

Plane: $$\mathrm{d}s^2 = R^2\left\{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\right\} \quad (9.6)$$

Hyperbolic Plane: $$\mathrm{d}s^2 = R^2\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2\,\mathrm{d}\theta^2\right\} \quad (9.18)$$

$\Longrightarrow$ All three metrics can be written as

$$\mathrm{d}s^2 = R^2\left\{\frac{\mathrm{d}r^2}{1-k\,r^2} + r^2\,\mathrm{d}\theta^2\right\} \quad (9.21)$$

where k defines the geometry:

$$k = \begin{cases} +1 & \text{spherical} \\ 0 & \text{planar} \\ -1 & \text{hyperbolic} \end{cases} \quad (9.22)$$

2D Metrics

For "spherical coordinates" we found:

Sphere: $$\mathrm{d}s^2 = R^2\left\{\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right\} \quad (9.14)$$

Plane: $$\mathrm{d}s^2 = R^2\left\{\mathrm{d}\theta^2 + \theta^2\mathrm{d}\phi^2\right\} \quad (9.6)$$

Hyperbolic Plane: $$\mathrm{d}s^2 = R^2\left\{\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2\right\} \quad (9.20)$$

$\Longrightarrow$ All three metrics can be written as

$$\mathrm{d}s^2 = R^2\left\{\mathrm{d}\theta^2 + S_k^2(\theta)\,\mathrm{d}\phi^2\right\} \quad (9.23)$$

where

$$S_k(\theta) = \begin{cases} \sin\theta & \text{for } k=+1 \\ \theta & \text{for } k=0 \\ \sinh\theta & \text{for } k=-1 \end{cases} \quad \text{and} \quad C_k(\theta) = \sqrt{1-kS_k^2(\theta)} = \begin{cases} \cos\theta & \text{for } k=+1 \\ 1 & \text{for } k=0 \\ \cosh\theta & \text{for } k=-1 \end{cases} \quad (9.24)$$

The cos-like analogue of S_k, C_k, will be needed later

Note that, compared to the earlier formulae, some coordinates have been renamed. This is confusing, but legal...

RW Metric

- Cosmological principle + expansion $\Longrightarrow \exists$ freely expanding cosmical coordinate system.
  - Observers =: fundamental observers
  - Time =: cosmic time

  This is the coordinate system in which the 3K radiation is isotropic, clocks can be synchronized, e.g., by adjusting time to the local density of the universe.

  $\Longrightarrow$ Metric has temporal and spatial part.

  This also follows directly from the equivalence principle.
- *Homogeneity and isotropy* $\Longrightarrow$ spatial part is spherically symmetric:

$$\mathrm{d}\psi^2 := \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \quad (9.25)$$

- *Expansion:* $\exists$ scale factor, $R(t) \Longrightarrow$ measure distances using comoving coordinates.

$\Longrightarrow$ metric looks like

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R^2(t)\left[f^2(r)\,\mathrm{d}r^2 + g^2(r)\,\mathrm{d}\psi^2\right] \quad (9.26)$$

where $f(r)$ and $g(r)$ are arbitrary.

RW Metric

Metrics of the form of eq. (9.26) are called Robertson-Walker (RW) metrics (introduced in 1935).

Previously studied by Friedmann and Lemaître. . .

One common choice is

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R^2(t)\left[\mathrm{d}r^2 + S_k^2(r)\,\mathrm{d}\psi^2\right] \qquad (9.27)$$

where

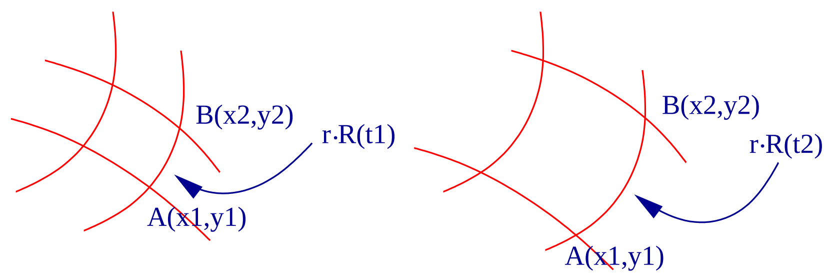
$R(t)$: scale factor, containing the physics
t: cosmic time
r, θ, ϕ: comoving coordinates (remember Eq. (9.25) ($\mathrm{d}\psi^2 := \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$)!)
k: defines curvature, integer

$S_k(r)$ was defined in Eq. (9.24).

Remark: θ and ϕ describe directions on sky, as seen from the arbitrary center of the coordinate system (=us), r can be interpreted as a radial coordinate.

RW Metric

The RW metric defines an universal coordinate system tied to expansion of space:



Scale factor $R(t)$ describes evolution of universe.

- r is called the comoving distance.
- $D(t) := r \cdot R(t)$ is called the proper distance,

(e.g., $r \cdot R(t)$ is measured in Mpc)

RW Metric

Other forms of the RW metric are also used:

1. Substitution $S_k(r) \longrightarrow r$ gives

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\psi^2\right\} \qquad (9.28)$$

(i.e., other definition of comoving radius r, which is still dimensionless).

2. A metric with a dimensionless scale factor,

$$a(t) := \frac{R(t)}{R(t_0)} = \frac{R(t)}{R_0} \qquad (9.29)$$

(where t_0 = today, i.e., $a(t_0) = 1$), gives

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left\{\mathrm{d}r^2 + \frac{S_k^2(R_0 r)}{R_0^2}\,\mathrm{d}\psi^2\right\} \qquad (9.30)$$

RW Metric

3. Using $a(t)$ *and* the substitution $S_k(r) \longrightarrow r$ is also possible:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left\{\frac{\mathrm{d}r^2}{1-k\cdot(R_0 r)^2} + r^2\,\mathrm{d}\psi^2\right\} \qquad (9.31)$$

The units of $R_0 r$ are Mpc $\Longrightarrow$ *Used for observations!*

4. Replace cosmic time, t, by conformal time, $\mathrm{d}\eta = \mathrm{d}t/R(t)$

$\Longrightarrow$ conformal metric,

$$\mathrm{d}s^2 = R^2(\eta)\left\{\mathrm{d}\eta^2 - \frac{\mathrm{d}r^2}{1-kr} - r^2\,\mathrm{d}\psi^2\right\} \qquad (9.32)$$

Theoretical importance of this metric: For $k = 0$, i.e., a flat space, the RW metric = Minkowski line element $\times\, R^2(\eta) \Longrightarrow$ Equivalence principle!

RW Metric

5. Finally, the metric can also be written in the isotropic form,

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \frac{R(t)}{1+(k/4)r^2}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\psi^2\right\} \tag{9.33}$$

Here, the term in $\{\ldots\}$ is just the line element of a 3d-sphere $\Longrightarrow$ isotropy!

Note: There are as many notations as authors, e.g., some use $a(t)$ where we use $R(t)$, etc. $\Longrightarrow$ Be careful!

Note 2: Local homogeneity and isotropy (i.e., within a Hubble radius, $r = c/H_0$), do not imply *global* homogeneity and isotropy $\Longrightarrow$ Cosmologies with a non-trivial topology are possible (e.g., also with more dimensions...).

Friedmann Equations, I

General relativistic approach: Insert metric into Einstein equation to obtain differential equation for $R(t)$:

Einstein equation:

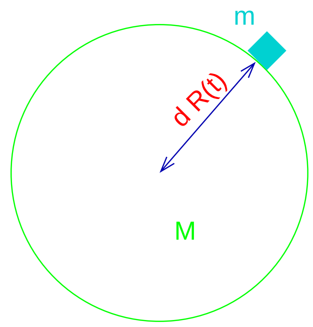
$$\underbrace{R_{\mu\nu} - \frac{1}{2}\mathscr{R}g_{\mu\nu}}_{G_{\mu\nu}} = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{9.34}$$

where

$g_{\mu\nu}$**:** Metric tensor ($\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$)
$R_{\mu\nu}$**:** Ricci tensor (function of $g_{\mu\nu}$)
$\mathscr{R}$**:** Ricci scalar (function of $g_{\mu\nu}$)
$G_{\mu\nu}$**:** Einstein tensor (function of $g_{\mu\nu}$)
$T_{\mu\nu}$**:** Stress-energy tensor, describing curvature of space due to fields present (matter, radiation,...)
Λ**:** Cosmological constant

$\Longrightarrow$ Messy, but doable

Friedmann Equations, II



Here, Newtonian derivation of Friedmann equations: Dynamics of a mass element on the surface of sphere of density $\rho(t)$ and comoving radius d, i.e., proper radius $d \cdot R(t)$ (McCrea, 1937)

Mass of sphere:

$$M = \frac{4\pi}{3}(d\,R)^3\rho(t) = \frac{4\pi}{3}d^3\rho_0 \quad \text{where} \quad \rho(t) = \frac{\rho_0}{R(t)^3} \tag{9.35}$$

Force on mass element:

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(d\,R(t)) = -\frac{GMm}{(dR(t))^2} = -\frac{4\pi G}{3}\frac{d\rho_0}{R^2(t)}m \tag{9.36}$$

Canceling $m \cdot d$ gives momentum equation:

$$\ddot{R}(t) = -\frac{4\pi G}{3}\frac{\rho_0}{R(t)^2} = -\frac{4\pi G}{3}\rho(t)R(t) \tag{9.37}$$

Multiplying Eq. (9.37) with $\dot{R}$ and integrating yields the energy equation:

$$\frac{1}{2}\dot{R}(t)^2 = +\frac{4\pi G}{3}\frac{\rho_0}{R(t)} + \text{const.} = +\frac{4\pi G}{3}\rho(t)R^2(t) + \text{const.} \tag{9.38}$$

where the constant can only be obtained from GR.

Friedmann Equations, III

Problems with the Newtonian derivation:

1. Cloud is implicitly assumed to have $r_{\text{cloud}} < \infty$
   (for $r_{\text{cloud}} \to \infty$ the force is undefined)
   $\Longrightarrow$ violates cosmological principle.
2. Particles move *through* space
   $\Longrightarrow$ $v > c$ possible
   $\Longrightarrow$ violates SRT.

Why do we get correct result?

GRT $\longrightarrow$ Newton for small scales and mass densities

Since universe is isotropic: scale invariance on Mpc scales

$\Longrightarrow$ Newton sufficient (classical limit of GR).

(In fact, point 1 above *does* hold in GR: Birkhoff's theorem).

Friedmann Equations, IV

The exact GR derivation of Friedmanns equation gives:

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \left[\frac{1}{3}\Lambda R\right]$$
$$\dot{R}^2 = +\frac{8\pi G\rho}{3}R^2 - kc^2 + \left[\frac{1}{3}\Lambda c^2 R^2\right] \quad (9.39)$$

Notes:

1. For $k = 0$: Eq. (9.39) $\longrightarrow$ Eq. (9.38).
2. k determines the curvature of space (and is *not* an integer here!).
3. The density, ρ, includes the contribution of all different kinds of energy (remember mass-energy equivalence!).
4. There is energy associated with the vacuum, parameterized by the parameter Λ.

In Eq. 9.49, it will be shown that the Hubble parameter can be expressed as $H(t) = \frac{\dot{R}}{R}$.

For $\Lambda = 0$, it evolves as:

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2(t) = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} \quad (9.40)$$

The Critical Density, I

Solving Eq. (9.40) for k:

$$\frac{R^2}{c}\left(\frac{8\pi G}{3}\rho - H^2\right) = k \quad (9.41)$$

$\Longrightarrow$ Sign of curvature parameter k only depends on density, ρ. With

$$\rho_c = \frac{3H^2}{8\pi G} \quad \text{and} \quad \Omega = \frac{\rho}{\rho_c} \quad (9.42)$$

it is easy to see that:

$\Omega > 1 \Longrightarrow k > 0 \Longrightarrow$ closed universe

$\Omega = 1 \Longrightarrow k = 0 \Longrightarrow$ flat universe

$\Omega < 1 \Longrightarrow k < 0 \Longrightarrow$ open universe

ρ_c is called the critical density.

For $\Omega \leq 1$ the universe will expand until ∞,

For $\Omega > 1$ we will see the "big crunch".

Current value of ρ_c: $\sim 1.67 \times 10^{-24}\,\mathrm{g\,cm^{-3}}$ (3...10 H-atoms $\mathrm{m^{-3}}$).

The Critical Density, II

Ω has a second order effect on the expansion:

Taylor series of $R(t)$ around $t = t_0$:

$$\frac{R(t)}{R(t_0)} = \frac{R(t_0)}{R(t_0)} + \frac{\dot{R}(t_0)}{R(t_0)}(t - t_0) + \frac{1}{2}\frac{\ddot{R}(t_0)}{R(t_0)}(t - t_0)^2 \quad (9.43)$$

The Friedmann equation Eq. (9.37) can be written

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho = -\frac{4\pi G}{3}\Omega\frac{3H^2}{8\pi G} = -\frac{\Omega H^2}{2} \quad (9.44)$$

Since $H(t) = \dot{R}/R$ (Eq. 9.49), Eq. (9.43) is

$$\frac{R(t)}{R(t_0)} = 1 + H_0(t - t_0) - \frac{1}{2}\frac{\Omega_0}{2}H_0^2(t - t_0)^2 \quad (9.45)$$

where $H_0 = H(t_0)$ and $\Omega_0 = \Omega(t_0)$.

The subscript 0 is often omitted in the case of Ω.

Often, Eq. (9.45) is written using the deceleration parameter:

$$q := \frac{\Omega}{2} = -\frac{\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)} \quad (9.46)$$

Hubble's Law

Hubble's Law follows from the variation of $R(t)$:

Small scales $\Longrightarrow$ Euclidean geometry. Then the proper distance between two observers is:

$$D(t) = d \cdot R(t) \quad (9.47)$$

where d: comoving distance.

Expansion $\Longrightarrow$ proper separation changes:

$$\frac{\Delta D}{\Delta t} = \frac{R(t+\Delta t)d - R(t)d}{\Delta t} \Longrightarrow \lim_{\Delta t \to 0} \Longrightarrow \quad v = \frac{\mathrm{d}D}{\mathrm{d}t} = \dot{R}\,d = \frac{\dot{R}}{R}D =: H\,D \quad (9.48)$$

$\Longrightarrow$ Identify local Hubble "constant" as

$$H = \frac{\dot{R}}{R} = \dot{a}(t) \quad (a(t) \text{ from Eq. 9.29, } a(\text{today}) = 1) \quad (9.49)$$

Since $R = R(t) \Longrightarrow H$ is time-dependent!

Redshift, I

The cosmological redshift is a consequence of the expansion of the universe:

The comoving distance is constant, thus in terms of the proper distance:

$$d = \frac{D(t=\text{today})}{R(t=\text{today})} = \frac{D(t)}{R(t)} = \text{const.} \tag{9.50}$$

Set $a(t) = R(t)/R(t=\text{today})$, then eq. (9.50) implies

$$\lambda_{\text{obs}} = \frac{\lambda_{\text{emit}}}{a_{\text{emit}}} \tag{9.51}$$

(λ_{obs}: observed wavelength, λ_{emit}: emitted wavelength)

Thus the observed redshift is

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} - 1 = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} - 1 \tag{9.52}$$

$$\Longrightarrow \quad 1 + z = \frac{1}{a_{\text{emit}}} = \frac{R(t=\text{today})}{R(t)} = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} \tag{9.53}$$

Light emitted at $z = 1$ was emitted when the universe was half as big as today!

z: measure for *relative size* of universe at time the observed light was emitted.

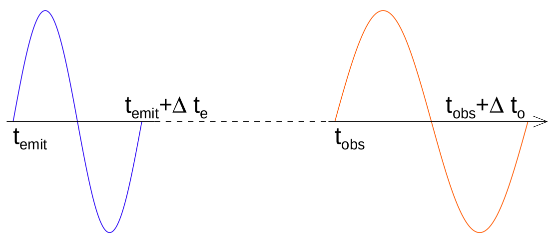
Note that the definition of H allows us to derive Hubble's relation for the case of small v, i.e., $v \ll c$. In this case, the red-shift is

$$z = \frac{v}{c} \quad \Longrightarrow \quad z = \frac{Hd}{c} \tag{9.54}$$

An alternative derivation of the cosmological redshift follows directly from general relativity, using the basic GR fact that for photons $\mathrm{d}s^2 = 0$. Inserting this into the metric, and assuming without loss of generality that $\mathrm{d}\psi^2 = 0$, one finds

$$0 = c^2\,\mathrm{d}t^2 - R^2(t)\,\mathrm{d}r^2 \quad \Longrightarrow \quad \mathrm{d}r = \pm\frac{c\,\mathrm{d}t}{R(t)} \tag{9.55}$$

Since photons travel forward, we choose the $+$-sign.



The *comoving* distance traveled by photons emitted at cosmic times t_{emit} and $t_{\text{emit}} + \Delta t_{\text{e}}$ is

$$r_1 = \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{c\,\mathrm{d}t}{R(t)} \quad \text{and} \quad r_2 = \int_{t_{\text{emit}}+\Delta t_{\text{e}}}^{t_{\text{obs}}+\Delta t_{\text{o}}} \frac{c\,\mathrm{d}t}{R(t)} \tag{9.56}$$

But the comoving distances are equal, $r_1 = r_2$! Therefore

$$0 = \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{c\,\mathrm{d}t}{R(t)} - \int_{t_{\text{emit}}+\Delta t_{\text{e}}}^{t_{\text{obs}}+\Delta t_{\text{o}}} \frac{c\,\mathrm{d}t}{R(t)} \tag{9.57}$$

$$= \int_{t_{\text{emit}}}^{t_{\text{emit}}+\Delta t_{\text{e}}} \frac{c\,\mathrm{d}t}{R(t)} - \int_{t_{\text{obs}}}^{t_{\text{obs}}+\Delta t_{\text{o}}} \frac{c\,\mathrm{d}t}{R(t)} \tag{9.58}$$

If Δt small $\Longrightarrow R(t) \approx$ const.:

$$= \frac{c\,\Delta t_{\text{e}}}{R(t_{\text{emit}})} - \frac{c\,\Delta t_{\text{o}}}{R(t_{\text{obs}})} \tag{9.59}$$

For a wave: $c\Delta t = \lambda$, such that

$$\frac{\lambda_{\text{emit}}}{R(t_{\text{emit}})} = \frac{\lambda_{\text{obs}}}{R(t_{\text{obs}})} \quad \Longleftrightarrow \quad \frac{\lambda_{\text{emit}}}{\lambda_{\text{obs}}} = \frac{R(t_{\text{emit}})}{R(t_{\text{obs}})} \tag{9.60}$$

From this equation it is straightforward to derive Eq. (9.52).

Redshift, II

Outside of the local universe: Eq. (9.53) only valid interpretation of z.

$\Longrightarrow$ It is common to interpret z as in special relativity:

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{9.61}$$

This is WRONG

Redshift is due to expansion of space, not due to motion of galaxy.

What *is* true is that z is accumulation of many infinitesimal red-shifts à la Eq. (9.54), see, e.g., Peacock (1999).

Time Dilatation

For light, $D = c\,\Delta t$. Then a consequence of Eq. (9.50) is

$$\frac{c\,\Delta t_{\text{emit}}}{R(t_{\text{emit}})} = \frac{c\,\Delta t_{\text{obs}}}{R(t_{\text{obs}})} \quad\Longrightarrow\quad \frac{\mathrm{d}t}{R} = \text{const.} \tag{9.59}$$

In other words:

$$\frac{\mathrm{d}t_{\text{obs}}}{\mathrm{d}t_{\text{emit}}} = \frac{R(t_{\text{obs}})}{R(t_{\text{emit}})} = 1 + z \tag{9.62}$$

$\Longrightarrow$ Time dilatation of events at large z.

This cosmological time dilatation has been observed in the light curves of supernova outbursts.

All other observables apart from z (e.g., number density $N(z)$, luminosity distance d_{L}, etc.) require explicit knowledge of $R(t)$

$\Longrightarrow$ Need to look at the dynamics of the universe.

Equation of state, I

Evolution of the universe determined by three different kinds of equation of state:

1. Matter: Normal (nonrelativistic) particles get diluted by expansion of the universe:

$$\rho_{\text{m}} \propto R^{-3} \tag{9.63}$$

   Matter is also often called dust by cosmologists.

2. Radiation: The energy density of radiation decreases because of volume expansion and because of the cosmological redshift (Eq. 9.60: $\lambda_{\text{obs}}/\lambda_{\text{emit}} = \nu_{\text{emit}}/\nu_{\text{obs}} = R(t_{\text{obs}})/R(t_{\text{emit}})$) such that

$$\rho_{\text{r}} \propto R^{-4} \tag{9.64}$$

3. Vacuum: The vacuum energy density (=Λ) is independent of R:

$$\rho_{\text{v}} = \text{const.} \tag{9.65}$$

Inserting these equations of state into the Friedmann equation and solving with the boundary condition $R(t=0) = 0$ then gives a specific world model.

Equation of state, II

Current scale factor is determined by H_0 and Ω_0:

Friedmann for $t = t_0$:

$$\dot{R}_0^2 - \frac{8\pi G}{3}\rho R_0^2 = -kc^2 \tag{9.66}$$

Insert Ω and note $H_0 = \dot{R}_0/R_0$

$$\Longleftrightarrow\quad H_0^2R_0^2 - H_0^2\Omega_0R_0^2 = -kc^2 \tag{9.67}$$

And therefore

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega - 1}} \tag{9.68}$$

For $\Omega \longrightarrow 0$, $R_0 \longrightarrow c/H_0$, the Hubble length.

For $\Omega = 1$, R_0 is arbitrary.

We now have everything we need to solve the Friedmann equation and determine the evolution of the universe for $k = 0, +1$, and -1.

$k = 0$, Matter dominated

For the matter dominated, flat case (the Einstein-de Sitter case), the Friedmann equation is

$$\dot{R}^2 - \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R^3}R^2 = 0 \tag{9.69}$$

For $k = 0$: $\Omega = 1$ and

$$\frac{8\pi G\rho_0}{3} = \Omega_0 H_0^2 R_0^3 = H_0^2 R_0^3 \tag{9.70}$$

Therefore, the Friedmann eq. is

$$\dot{R}^2 - \frac{H_0^2R_0^3}{R} = 0 \quad\Longrightarrow\quad \frac{\mathrm{d}R}{\mathrm{d}t} = H_0R_0^{3/2}R^{-1/2} \tag{9.71}$$

Separation of variables and setting $R(0) = 0$,

$$\int_0^{R(t)} R^{1/2}\,\mathrm{d}R = H_0R_0^{3/2}t \quad\Longrightarrow\quad \frac{2}{3}R^{3/2}(t) = H_0R_0^{3/2}t \quad\Longrightarrow\quad R(t) = R_0\left(\frac{3H_0}{2}\,t\right)^{2/3} \tag{9.72}$$

Therefore, for $k = 0$, the universe expands until ∞, its current age ($R(t_0) = R_0$) is given by

$$t_0 = \frac{2}{3H_0} \tag{9.73}$$

Reminder: The Hubble-Time is $H_0^{-1} = 9.78\,\mathrm{Gyr}/h$.

For the matter dominated, closed case, Friedmann's equation is

$$\dot{R}^2 - \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R} = -c^2 \iff \dot{R}^2 - \frac{H_0^2 R_0^3 \Omega_0}{R} = -c^2 \quad (9.74)$$

Inserting R_0 from Eq. (9.68) gives

$$\dot{R}^2 - \frac{H_0^2 c^3 \Omega_0}{H_0^3 (\Omega - 1)^{3/2}} \frac{1}{R} = -c^2 \quad (9.75)$$

which is equivalent to

$$\frac{\mathrm{d}R}{\mathrm{d}t} = c\left(\frac{\xi}{R} - 1\right)^{1/2} \quad \text{with} \quad \xi = \frac{c}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \quad (9.76)$$

With the boundary condition $R(0) = 0$, separation of variables gives

$$ct = \int_0^{R(t)} \frac{\mathrm{d}R}{(\xi/R - 1)^{1/2}} = \int_0^{R(t)} \frac{\sqrt{R}\,\mathrm{d}R}{(\xi - R)^{1/2}} \quad (9.77)$$

Integration by substitution gives the "cycloid solution"

$$R = \xi \sin^2\frac{\theta}{2} = \frac{\xi}{2}(1 - \cos\theta) \quad \text{and} \quad ct = \frac{\xi}{2}(\theta - \sin\theta) \quad (9.78)$$

where θ is an implicit parameter.

The age of the universe, t_0, is obtained by solving

$$R_0 = \frac{c}{H_0(\Omega_0 - 1)^{1/2}} = \frac{\xi}{2}(1 - \cos\theta_0) = \frac{1}{2}\frac{c}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(1 - \cos\theta_0) \quad (9.79)$$

(remember Eq. 9.68!). Therefore

$$\cos\theta_0 = \frac{2 - \Omega_0}{\Omega_0} \iff \sin\theta_0 = \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1} \quad (9.80)$$

Inserting this into Eq. (9.78) gives

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left[\arccos\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1}\right] \quad (9.81)$$

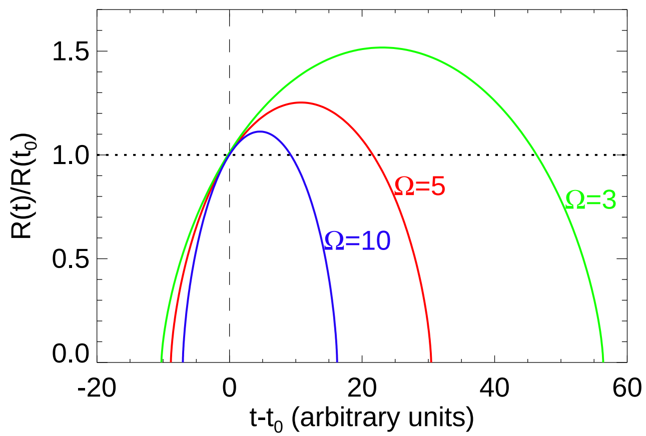
The cycloid solution shows that for $\Omega > 1$, the universe has a finite lifetime, i.e., it expands to a maximum and then becomes smaller and dies in a "big crunch". The max. expansion occurs at $\theta = \pi$, with a maximum scale factor of

$$R_{\text{max}} = \xi = \frac{c}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \quad (9.82)$$

The big crunch will happen at $\theta = 2\pi$, such that the lifetime of the closed universe is

$$t_{\text{life}} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \quad (9.83)$$

$k = +1$, Matter dominated, I



For the closed universe, one finds

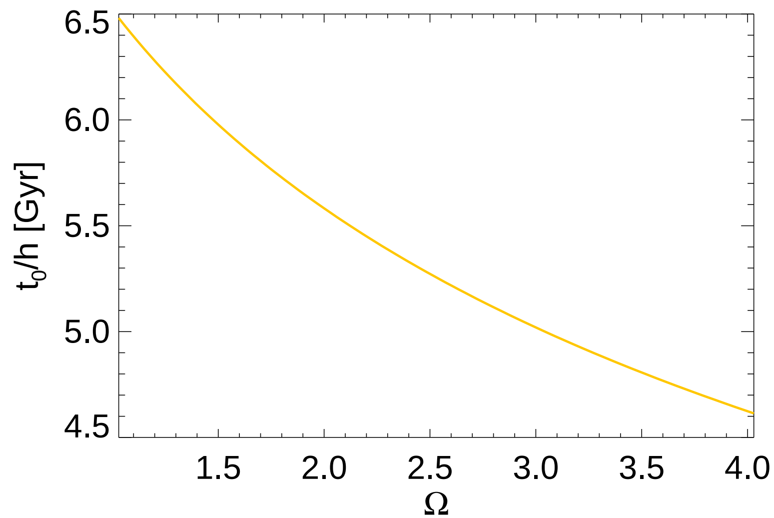
$$R = \frac{\xi}{2}(1 - \cos\theta) \qquad ct = \frac{\xi}{2}(\theta - \sin\theta) \quad (9.78)$$

Note that R is a cyclic function

$\Longrightarrow$ The closed universe has a finite lifetime, given by

$$t_{\text{life}} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \quad (9.83)$$

$k = +1$, Matter dominated, II



Age of a closed and matter dominated universe.

$k = -1$, Matter dominated, I

Finally, the matter dominated, open case. This case is very similar to the case of $k = +1$:

For $k = -1$, the Friedmann equation becomes

$$\frac{\mathrm{d}R}{\mathrm{d}t} = c\left(\frac{\zeta}{R} + 1\right)^{1/2} \tag{9.84}$$

where

$$\zeta = \frac{c}{H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}} \tag{9.85}$$

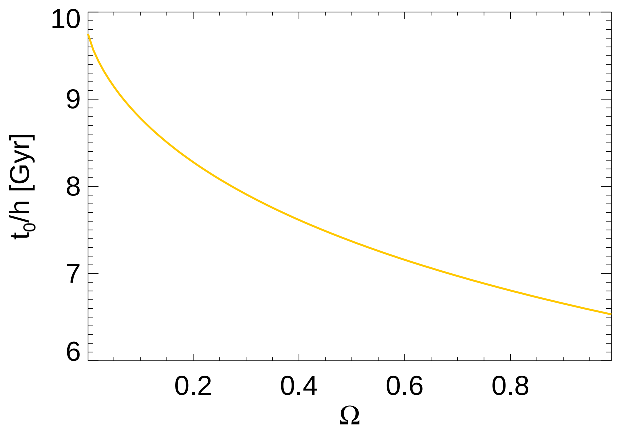
Separation of variables gives after a little bit of algebra

$$\begin{aligned} R &= \frac{\zeta}{2}(\cosh\theta - 1) \\ ct &= \frac{\zeta}{2}(\sinh\theta - 1) \end{aligned} \tag{9.86}$$

where the integration was again performed by substitution.

Note: θ here has *nothing* to do with the coordinate angle θ!

$k = -1$, Matter dominated, II



To obtain the age of the universe, note that at the present time,

$$\begin{aligned} \cosh\theta_0 &= \frac{2-\Omega_0}{\Omega_0} \\ \sinh\theta_0 &= \frac{2}{\Omega_0}\sqrt{1-\Omega_0} \end{aligned} \tag{9.87}$$

(identical derivation as that leading to Eq. 9.79)

therefore,

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}} \cdot \left\{\frac{2}{\Omega_0}\sqrt{1-\Omega_0} - \ln\left(\frac{2-\Omega_0+2\sqrt{1-\Omega_0}}{\Omega_0}\right)\right\} \tag{9.88}$$

Summary

For the matter dominated case, our results from Eqs. (9.78), and (9.86) can be written with the functions S_k and C_k (Eq. 9.24) in form of the cycloid solution:

$$\begin{aligned} R &= k\mathscr{R}\,(1 - C_k(\theta)) \\ ct &= k\mathscr{R}\,(\theta - S_k(\theta)) \end{aligned} \tag{9.89}$$

with

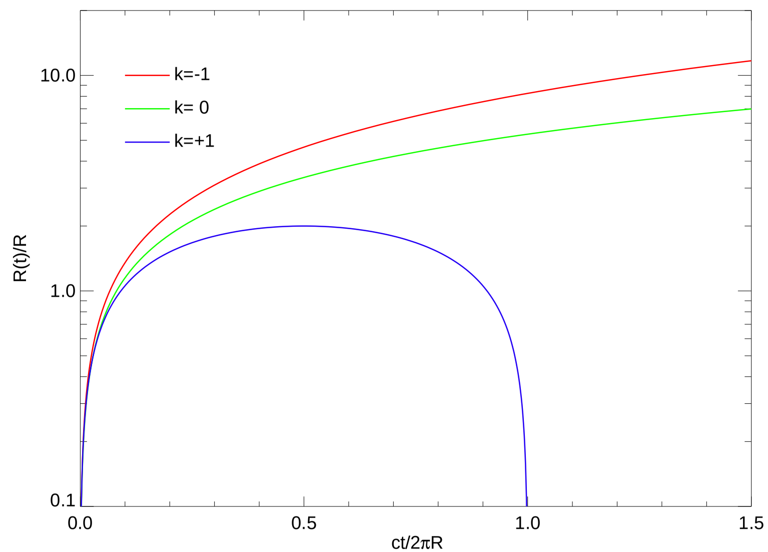
$$S_k(\theta) = \begin{cases} \sin\theta \\ \theta \\ \sinh\theta \end{cases} \quad \text{and} \quad C_k(\theta) = \begin{cases} \cos\theta & \text{for } k = +1 \\ 1 & \text{for } k = 0 \\ \cosh\theta & \text{for } k = -1 \end{cases} \tag{9.24}$$

and where the characteristic radius, $\mathscr{R}$, is given by

$$\mathscr{R} = \frac{c}{H_0}\frac{\Omega_0/2}{(k\,(\Omega_0 - 1))^{3/2}} \tag{9.90}$$

Notes:

1. Eq. (9.89) can also be derived as the result of the Newtonian collapse/expansion of a spherical mass distribution.
2. θ is called the development angle, it is equal to the *conformal time* (Eq. (9.32)).



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