



The Hot Big Bang

CMBR

Assumption: Early universe was hot and dense

⇒ Equilibrium between matter and radiation.

Generation of radiation, e.g., in pair equilibrium,



Equilibrium with electrons, e.g., via Compton scattering:



where the electrons are linked to protons via Coulomb interaction.

Once density low and temperature below photoionization for Hydrogen,



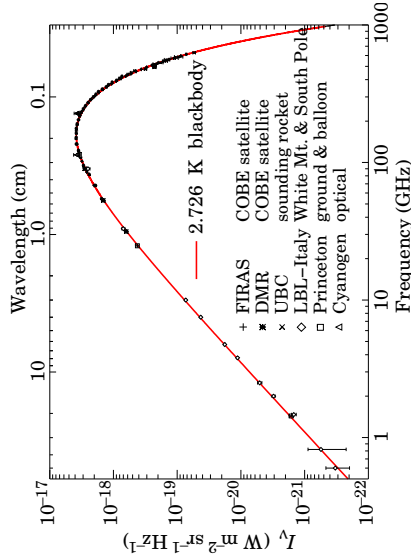
Decoupling of radiation and matter ⇒ Adiabatic cooling of photon field.

Proof for these assumptions, and lots of gory details: this and the next few lectures!

Motivation



CMBR



(after Smoot, 1997, Fig. 1)

The CMBR spectrum is fully consistent with a pure Planckian with temperature $T_{\text{CMBR}} = 2.728 \pm 0.004 \text{ K}$: a relic of the hot big bang.

Motivation



Motivation



CMBR

Reminder: Planck formula for energy density of photons:

$$B_\lambda = \frac{du}{d\lambda} = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp(hc/k_B T \lambda) - 1} \quad (11.4)$$

(units: $\text{erg cm}^{-3} \text{ \AA}^{-1}$), where

$k_B = 1.38 \times 10^{-16} \text{ erg K}^{-1}$ (Boltzmann) and $h = 6.625 \times 10^{-27} \text{ ergs}$ (Planck)

(11.5)

For $\lambda \gg hc/k_B T$: Rayleigh-Jeans formula:

$$B_\lambda \sim \frac{8\pi k_B T}{\lambda^4} \quad (11.6)$$

(classical case, diverges for $\lambda \rightarrow 0$, “Jeans catastrophe”).

The wavelength of maximum emission is given by Wien’s displacement law:

$$\lambda_{\text{max}} = 0.201 \frac{hc}{k_B T} \quad (11.7)$$

Motivation

**CMBR**

The total energy density of the CMB is obtained by integration:

$$u = \int_0^\infty B_\lambda d\lambda = \frac{8\pi^5 (kT)^4}{15h^3 c^3} = \frac{4\sigma_{\text{SB}} T^4}{c} = a_{\text{rad}} T^4 \quad (11.8)$$

where

$$\sigma_{\text{SB}} = 5.670 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^{-4} \quad \text{Stefan-Boltzmann} \quad (11.9)$$

$$a_{\text{rad}} = 7.566 \times 10^{-15} \text{ erg cm}^{-2} \text{ K}^{-4} \text{ s}^{-1} \quad \text{radiation density constant} \quad (11.10)$$

Since the energy of a photon is $E_\gamma = h\nu = hc/\lambda$, the total number density of photons is

$$n = \int_0^\infty \frac{B_\lambda d\lambda}{hc/\lambda} = 20.28 T^3 \text{ photons cm}^{-3} \quad (11.11)$$

Thus, for today's CMBR:

$$n_{\text{CMBR}} = 400 \text{ photons cm}^{-3} \quad (11.12)$$

Motivation



For the CMBR today:

$$n_{\text{CMBR}} = 400 \text{ photons cm}^{-3} \quad (11.12)$$

Compare that to gravitating matter (protons for now).

⇒ critical density:

$$\rho_c = \frac{3H^2}{8\pi G} = 1.88 \times 10^{-29} h^2 \text{ g cm}^{-3} \doteq 1.13 \times 10^{-5} h^2 \text{ protons cm}^{-3} \quad (7.42)$$

since $m_p = 1.67 \times 10^{-24} \text{ g}$.

⇒ photons dominate the particle number:

$$\frac{n_{\text{CMBR}}}{n_{\text{baryons}}} = \frac{3.54 \times 10^7}{\Omega h^2} \quad (11.13)$$

⇒ baryons dominate the energy density:

$$\frac{u_{\text{CMBR}}}{u_{\text{baryons}}} = \frac{a_{\text{rad}} T^4}{\Omega \rho_c c^2} = \frac{4.20 \times 10^{-13}}{1.69 \times 10^{-8} \Omega h^2} = \frac{1}{40260 \Omega h^2} \quad (11.14)$$

That's why we talk about the **matter dominated universe**.

Motivation

**CMBR**

But: The Universe was not always matter dominated:

Remember the scaling laws for the (energy) density of matter and radiation:

$$\begin{aligned} \rho_m &\propto R^{-3} \\ \rho_r &\propto R^{-4} \end{aligned} \implies \frac{\rho_r}{\rho_m} \propto \frac{1}{R} \quad (7.63, 7.64)$$

⇒ Photons dominate for large z , i.e., early in the universe!

Since $1+z = R_0/R$ (Eq. 7.53), matter-radiation equality was at

$$1+z_{\text{eq}} = 40260 \Omega h^2 \quad (11.15)$$

(for $h = 0.75$, $1+z_{\text{eq}} = 22650$)

The above definition of z_{eq} is not entirely correct: neutrino background, which increases the background energy density, is ignored ($u_\nu \sim 68\%$ of u_r , see later).

Formally, matter-radiation equality defined from $n_{\text{baryons}} = n_{\text{relativistic particles}}$, giving

$$1+z_{\text{eq}} = 23900 \Omega h^2 \quad (11.16)$$

(for $h = 0.75$, $1+z_{\text{eq}} = 13440$).

Motivation



What happened to the temperature of the CMBR?

Compare CMBR spectrum today with earlier times.

(Differential) Energy density in $[\lambda, \lambda + d\lambda]$:

$$du = B_\lambda d\lambda \quad (11.17)$$

Cosmological redshift:

$$\frac{\lambda'}{\lambda} = \frac{R'}{R} = \frac{1}{1+z} = a \quad (7.60)$$

Taking the expansion into account:

$$\begin{aligned} du' &= \frac{du}{a^4} = \frac{8\pi hc}{a^4 \lambda^5} \frac{d\lambda}{\exp(hc/kT\lambda) - 1} = \frac{8\pi hc}{a^5 \lambda^5} \frac{a d\lambda}{\exp(hc/kT\lambda) - 1} \\ &= \frac{8\pi hc}{\lambda'^5} \frac{d\lambda'}{\exp(hca/kT\lambda') - 1} = B_{\lambda'}(T/a) d\lambda' \quad (11.18) \end{aligned}$$

Therefore, the Planckian remains a Planckian, and the temperature of the CMBR scales as

$$T(z) = (1+z)T_0 \quad (11.19)$$

The early universe was hot ⇒ Hot Big Bang Model!

Motivation



Overview

$a(t)$	t since BB	T [K]	ρ_{matter} [g cm ⁻³]	Major Events
	10^{-42}	10^{30}		Planck era, "begin of physics"
	$10^{-40} \dots 10^{-30}$	10^{25}		Inflation?
10^{-13}	$\sim 10^{-5}$ s	$\sim 10^{13}$	$\sim 10^9$	generation of p-p ⁺ , and baryon anti-baryon pairs from radiation background
3×10^{-9}	1 min	10^{10}	0.03	generation of e ⁺ -e ⁻ pairs out of radiation background
10^{-9}	10 min	3×10^9	10^{-3}	nucleosynthesis
$10^{-4} \dots 10^{-3}$	$10^{6 \dots 7}$ yr	$10^{3 \dots 4}$	$10^{-21 \dots -18}$	End of radiation dominated epoch
7×10^{-4}	10^7 yr	4000	10^{-20}	Hydrogen recombines, decoupling of matter and radiation
1	15×10^9 yr	3	10^{-30}	now

Overview



Density in early universe is very high.

Physical processes (e.g., photon-photon pair creation, electron-positron annihilation etc.) all have reaction rates

$$\Gamma \propto n\sigma v \quad (11.20)$$

where

n : number density (cm⁻³)

σ : interaction cross-section (cm²)

v : velocity (cm s⁻¹)

Thermodynamic equilibrium is reached if reaction rate much faster than "changes" in the system,

$$\Gamma \gg H \quad (11.21)$$

Where the Hubble parameter, H , is a good measure for (typical timescale of the Universe)⁻¹.

If thermodynamic equilibrium holds, then we can assume evolution of universe as sequence of states of local thermodynamic equilibrium, and use standard thermodynamics.

Before looking at real universe, first need to derive certain useful formulae from relativistic thermodynamics.



Thermodynamics

For ideal gases, thermodynamics shows that number density $f(\mathbf{p}) d^3p$ of particles with momentum in $[p, p + dp]$ is given by

$$f(\mathbf{p}) = \frac{1}{\exp((E - \mu)/k_B T) + a} \quad (11.22)$$

where

$$a = \begin{cases} +1 & \text{: Fermions (spin=1/2, 3/2, \dots)} \\ -1 & \text{: Bosons (spin=1, 2, \dots)} \\ 0 & \text{: Maxwell-Boltzmann} \end{cases}$$

and where the energy includes the rest-mass:

$$E^2 = |\mathbf{p}|^2 c^2 + m^2 c^4 \quad (11.23)$$

μ is called the "chemical potential". It is preserved in chemical equilibrium:

$$i + j \leftrightarrow k + l \implies \mu_i + \mu_j = \mu_k + \mu_l \quad (11.24)$$

photons: multi-photon processes exist $\implies \mu_\gamma = 0$.

particles in thermal equilibrium: $\mu = 0$ as well, because of the first law of thermodynamics,

$$dE = T dS - P dV + \mu dN \quad (11.25)$$

which, in equilibrium, is stationary with respect to changes in particle number N .

Big Bang Thermodynamics



Thermodynamics

In addition to number density: different particles have internal degrees of freedom, g .

Examples:

photons: two polarization states $\implies g = 2$

neutrinos: one polarization state $\implies g = 1$

electrons, positrons: spin=1/2 $\implies g = 2$

Knowing g and $f(p)$, it is possible to calculate interesting quantities:

$$\text{particle number density: } n = \frac{g}{(2\pi\hbar)^3} \int f(\mathbf{p}) d^3p \quad (11.26)$$

$$\text{energy density: } u = \rho c^2 = \frac{g}{(2\pi\hbar)^3} \int E(\mathbf{p}) f(\mathbf{p}) d^3p \quad (11.27)$$

To calculate the pressure, remember that kinetic theory shows:

$$P = \frac{n}{3} \langle pv \rangle = \frac{n}{3} \left\langle \frac{p^2 c^2}{E} \right\rangle \quad (11.28)$$

such that

$$P = \frac{g}{(2\pi\hbar)^3} \int \frac{p^2 c^2}{3E} f(\mathbf{p}) d^3p \quad (11.29)$$



Thermodynamics

Generally, we are interested in knowing n , u , and P in two limiting cases:

1. the ultra-relativistic limit, where $k_B T \gg mc^2$, i.e., kinetic energy dominates the rest-mass
2. the non-relativistic limit, where $k_B T \ll mc^2$

Transitions between these limits (i.e., what happens during “cooling”) are usually much more complicated \implies ignore...

Big Bang Thermodynamics

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11-13

To derive the number density, the energy density, and the equation of state, note that Eq. (11.23) shows

$$E = \sqrt{p^2 c^2 + m^2 c^4} \quad (11.23)$$

such that

$$p = \sqrt{E^2 - m^2 c^4} / c \quad (11.30)$$

Therefore

$$\frac{dE}{dp} = \frac{pc^2}{\sqrt{p^2 c^2 + m^2 c^4}} \quad (11.31)$$

from which it follows that

$$E \, dE = pc^2 \, dp \quad (11.32)$$

Thus the following holds

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d^3 p = \int_0^{\infty} 4\pi p^2 \, dp = \int_{mc^2}^{\infty} \frac{4\pi}{mc^2} (E^2 - m^2 c^4)^{1/2} E \, dE \quad (11.33)$$

Going to a system of units where

$$c = k_B = \hbar = 1 \quad (11.34)$$

to save me some typing, substitute these equations into Eqs. (11.26)–(11.29) to find

$$n = \frac{g}{2\pi^2} \int_{J_m}^{\infty} \frac{(E^2 - m^2)^{1/2} E \, dE}{\exp((E - \mu)/T) \pm 1} \quad (11.35)$$

$$\rho = \frac{g}{2\pi^2} \int_{J_m}^{\infty} \frac{(E^2 - m^2)^{1/2} E^2 \, dE}{\exp((E - \mu)/T) \pm 1} \quad (11.36)$$

$$P = \frac{g}{6\pi^2} \int_{J_m}^{\infty} \frac{(E^2 - m^2)^{3/2} \, dE}{\exp((E - \mu)/T) \pm 1} \quad (11.37)$$

which can in some limiting cases be expressed in a closed form (Kolb & Turner, 1990, eq. 3.52 ff., see following vignettes).



Thermodynamics

In the ultra-relativistic limit, $k_B T \gg mc^2$, and assuming $\mu = 0$,

$$n = \begin{cases} \frac{\zeta(3)}{\pi^2} g \left(\frac{k_B T}{\hbar c}\right)^3 & \text{Bosons} \\ \frac{3}{4} \frac{\zeta(3)}{\pi^2} g \left(\frac{k_B T}{\hbar c}\right)^3 & \text{Fermions} \end{cases} \quad (11.38)$$

$$u = \begin{cases} \frac{\pi^2}{30} g k_B T \left(\frac{k_B T}{\hbar c}\right)^3 & \text{Bosons} \\ \frac{7}{8} \frac{\pi^2}{30} g k_B T \left(\frac{k_B T}{\hbar c}\right)^3 & \text{Fermions} \end{cases} \quad (11.39)$$

$$P = \rho c^2 / 3 = u / 3 \quad (11.40)$$

where $\zeta(3) = 1.202\dots$, and $\zeta(s)$ is Riemann's zeta-function (see handout, Eq. 11.48).

Eq. (11.40) is a simple result of the fact that in the relativistic limit, $E \sim pc$. Inserting this and $v = c$ into Eq. (11.28) gives the desired result.

As expected, we find the T^4 proportionality from the Stefan Boltzmann law!

Big Bang Thermodynamics

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11-14

Obtaining the previous formulae is an exercise in special functions. For example, the $T \gg m$, $T \gg \mu$ case for ρ for Bosons (Eq. 11.33) is obtained as follows (setting $c = k_B = \hbar = 1$):

$$\rho_{\text{boson}} = \frac{g}{2\pi^2} \int_{J_m}^{\infty} \frac{(E^2 - m^2)^{1/2} E^2 \, dE}{\exp((E - \mu)/T) \pm 1} \quad (11.41)$$

because of $T \gg \mu$

$$\approx \frac{g}{2\pi^2} \int_{J_m}^{\infty} \frac{(E^2 - m^2)^{1/2} E^2 \, dE}{\exp(E/T) \pm 1} \quad (11.42)$$

for Bosons, choose -1 , and substitute $x = E/T$:

$$= \frac{g}{2\pi^2} \int_{J_m/T}^{\infty} \frac{(x^2 T^2 - m^2)^{1/2} x^2 T^3 \, dx}{\exp(x) - 1} \quad (11.43)$$

Since $T \gg m$,

$$\approx \frac{g}{2\pi^2} \int_0^{\infty} \frac{x^3 T^4 \, dx}{\exp(x) - 1} \quad (11.44)$$

$$= \frac{g T^4}{2\pi^2} \int_0^{\infty} \frac{x^3 \, dx}{\exp(x) - 1} \quad (11.45)$$

$$= \frac{g T^4}{2\pi^2} \cdot 6\zeta(4) \quad (11.46)$$

$$= \frac{7}{8} \frac{g T^4}{30} \quad (11.47)$$

where $\zeta(s)$ is Riemann's zeta-function, which is defined by (Abramowitz & Stegun, 1964)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{\exp(x) - 1} \, dx \quad \text{for } \Re s > 1 \quad (11.48)$$

where $\Gamma(x)$ is the Gamma-function. Note that $\zeta(4) = \pi^4/90$.



Equation of State

Pressure of ultra-relativistic particles \gg Pressure of nonrelativistic particles

\implies Nonrelativistic particles unimportant for equation of state.

For relativistic particles:

$$u_{\text{bosons}} = \frac{\pi^2}{30} g k_B T \left(\frac{k_B T}{\hbar c} \right)^3 \quad \text{and} \quad u_{\text{fermions}} = \frac{7}{8} u_{\text{bosons}} \quad (11.39)$$

\implies Total energy density for mixture of particles is sum over all particles, written as

$$u = g_* \cdot \frac{\pi^2}{30} k_B T \left(\frac{k_B T}{\hbar c} \right)^3 \quad (11.54)$$

where the effective degeneracy factor

$$g_* = \sum_{\text{bosons}} g_B \left(\frac{T_B}{T} \right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_F \left(\frac{T_F}{T} \right)^4 \quad (11.55)$$

g_* counts total number of internal degrees of freedom of all relativistic bosonic and fermionic species, i.e., all relativistic particles which are in thermodynamic equilibrium

The pressure is obtained from Eq. (11.54) as $P = u/3$.

Big Bang Thermodynamics



Thermodynamics

In the non-relativistic limit: $k_B T \ll mc^2$

\implies we can ignore the ± 1 term in the denominator

\implies Same formulae for Bosons and Fermions!

$$n = \frac{2g}{(2\pi\hbar)^3} (2\pi m k_B T)^{3/2} e^{-mc^2/k_B T} \quad (11.51)$$

$$u = n m c^2 \quad (11.52)$$

$$P = n k_B T \quad (11.53)$$

Therefore:

- energy density dominated by rest-mass ($\rho = u/c^2 = mn$)
- $P \ll \rho c^2/3$, i.e., *much* smaller than for relativistic particles
- \implies Particle pressure is only important for relativistic particles

Obviously, relativistic particles with $m = 0$ (or very close to 0) will never become nonrelativistic. Still, they can "decouple" from the rest of the universe when their interaction rates go to 0.

Big Bang Thermodynamics

Knowing the equation of state, we can now use the Friedmann equations to determine the early evolution of the universe.

Friedmann:

$$\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 \quad (7.39)$$

or, dividing by R^2 ,

$$\frac{\dot{R}^2}{R^2} = H(t)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{R^2} \quad (7.40)$$

But: The early universe is dominated by relativistic particles

$\implies \rho \propto R^{-4}$

\implies Density-term dominates

\implies we can set $k = 0$.

Early universe is asymptotically flat!

This will prove to be one of the most crucial problems of modern cosmology...

Early Universe

For *Fermions*, everything is the same except for that we now have to choose the + sign. The equivalent of Eq. (11.45) is then

$$P_{\text{fermi}} = \frac{gT^4}{2\pi^2} \int_0^\infty \frac{x^3 dx}{\exp(x) + 1} \quad (11.49)$$

Now we can make use of formula 3.411.3 of Gradshteyn & Ryzhik (1981).

$$\int_0^\infty \frac{x^{\nu-1} dx}{\exp(\mu x) + 1} = \frac{1}{\mu^\nu} (1 - 2^{-\nu}) \Gamma(\nu) \zeta(\nu) \quad \text{for } \Re \nu, \mu, \nu > 1 \quad (11.50)$$

to see where the additional factor of $7/8$ in Eq. (11.39) comes from.



Early Expansion

The Friedmann equations show the evolution of the early Universe:

Since $\rho \propto R^{-4}$ (relativistic background),

$$\rho = \rho_0 \left(\frac{R_0}{R}\right)^4 \tag{11.56}$$

Friedmann's equation then is

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G \rho}{3}} R = \sqrt{\frac{8\pi G \rho_0 R_0^2}{3}} \frac{1}{R} \Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G \rho_0}{3}} \frac{1}{a} =: \xi a^{-1} \tag{11.57}$$

where $a = R/R_0$.

Separation of variables gives

$$\int_0^{a(t)} a \, da = \int_0^t \xi \, dt \Rightarrow a(t) = \xi^{1/2} \cdot t^{1/2} \tag{11.58}$$

Therefore, for $k = 0$ the Hubble constant evolves as

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{2t} \tag{11.59}$$

Early Universe

Early Expansion

Using the equation of state from Eq. 11.54, we can relate $H(t)$ to the temperature.

Inserting Eq. 11.54) into Friedmann's equation gives

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2(t) = \frac{8\pi G \rho}{3} = \frac{8\pi G}{3} g_* \frac{\pi^2 (k_B T)^4}{30 (\hbar c)^3} = \frac{4\pi^3 G}{45 (\hbar c)^3} g_* (k_B T)^4 \tag{11.60}$$

and with Eq. 11.59:

$$H(t) = \left(\frac{4\pi^3 G}{45 (\hbar c)^3}\right)^{1/2} g_*^{1/2} (k_B T)^2 = \frac{1}{2t} \tag{11.61}$$

Solving for t gives the time-temperature relationship:

$$t = \left(\frac{45 (\hbar c)^3}{16\pi^3 G}\right)^{1/2} \frac{1}{g_*^{1/2}} \frac{1}{(k_B T)^2} \tag{11.62}$$

Inserting all constants and converting to more useful units gives

$$t = \frac{2.4 \text{ s}}{g_*^{1/2}} \left(\frac{k_B T}{1 \text{ MeV}}\right)^{-2} \tag{11.63}$$

... one of the most useful equations for the early universe.

Early Universe

Elementary Particles

Behavior of universe depends on g_* \implies Strong dependency on elementary particle physics.

Generally, particles present when energy in other particles allows generation of particle-antiparticle pairs, i.e., when $k_B T \gtrsim mc^2$ (threshold temperature)

Current particle physics provides the following picture:

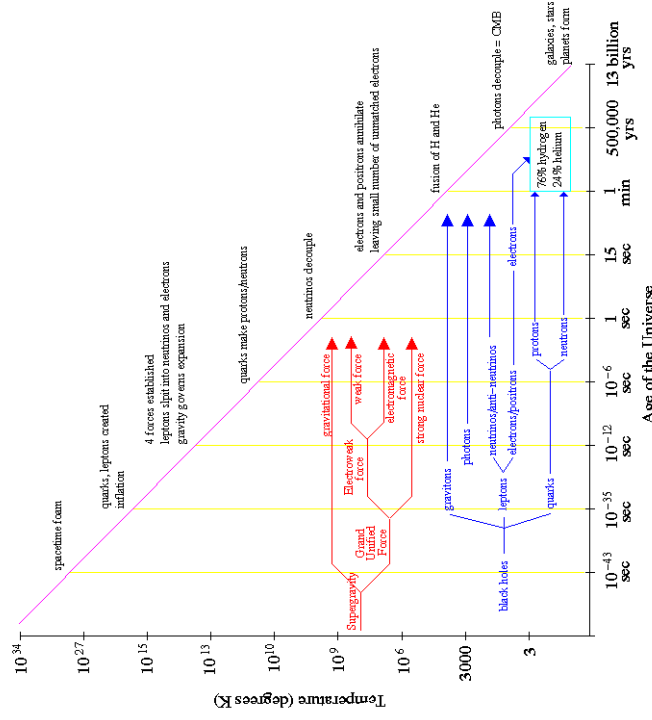
Temp.	New Particles	$4g_*$
$k_B T < m_e c^2$	γ 's and ν 's	29
$m_e c^2 < k_B T < m_\mu c^2$	e^\pm	43
$m_\mu c^2 < k_B T < m_\pi c^2$	μ^\pm	57
$m_\pi c^2 < k_B T < k_B T_c$	π^\pm	69
$k_B T_c < k_B T < m_{\text{strange}} c^2$	$-\pi^+ - \pi^-$, u, \bar{u}, d, \bar{d} , gluons	205
$m_s c^2 < k_B T < m_{\text{charm}} c^2$	s, \bar{s}	247
$m_c c^2 < k_B T < m_\tau c^2$	c, \bar{c}	289
$m_\tau c^2 < k_B T < m_{\text{bottom}} c^2$	τ^\pm	303
$m_b c^2 < k_B T < m_{W,Z} c^2$	b, \bar{b}	345
$m_{W,Z} c^2 < k_B T < m_{\text{top}} c^2$	W^\pm, Z	381
$m_t c^2 < k_B T < m_{\text{Higgs}} c^2$	t, \bar{t}	423
$m_{\text{Higgs}} c^2 < k_B T$	H^0	427

$k_B T_c$: energy of confinement-deconfinement for transitions quarks \implies hadrons, somewhere between 150 MeV and 400 MeV.

Example: photons (2 polarization states, i.e., $g = 2$) and three species of neutrinos ($g = 1$, but with distinguishable anti-particles) $\implies g_* = 2 + (7/8) \cdot 2 \cdot 3 = 58/8 = 29/4$.

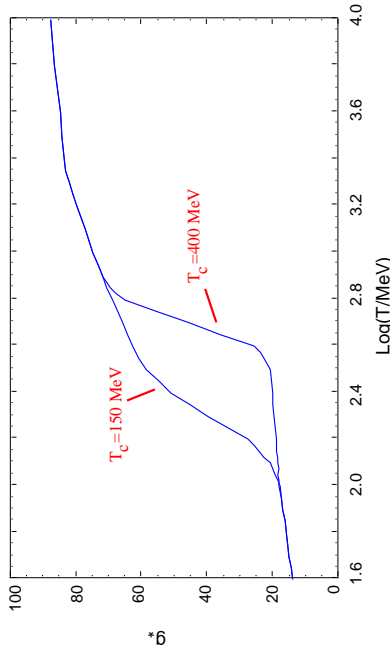
(Olive, 1999, Tab. 1)

Early Universe





Elementary Particles



(Olive, 1999, Fig. 1)

Will now consider times when only Neutrinos and Electron/Positrons present (after baryogenesis, see next lecture for that).

Early Universe



Interlude

Previous (abstract) formulae allow to estimate quantities like

1. The existence and energy of primordial neutrinos,
2. The formation of neutrons,
3. The formation of heavier elements.

Detailed computations require solving nonlinear differential equations

⇒ difficult, only numerically possible.

Essentially, need to self-consistently solve Boltzmann equation in expanding universe for evolution of phase space density with time, using the correct QCD/QED reaction rates ⇒ too complicated (at least for me...).

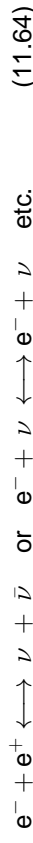
Will use **approximate analytical way** here, which gives surprisingly exact answers.

Neutrinos



Neutrinos

Neutrino equilibrium caused by weak interactions such as



Reaction rate for these processes:

$$\Gamma = n \langle \sigma v \rangle \quad (11.65)$$

where the thermally averaged interaction cross-section is

$$\langle \sigma v \rangle \approx \left\langle \frac{\alpha^2 p}{m_W^4} \cdot p \right\rangle \sim 10^{-2} \frac{(k_B T)^2}{m_W^4} \quad (11.66)$$

m_W : mass of W-boson (exchange particle of weak interaction), $\alpha \approx 1/137$: fine structure constant.

But in the ultra-relativistic limit, $n \propto T^3$ (Eq. 11.38), such that

$$\Gamma_{\text{weak}} \propto \frac{\alpha^2 T^5}{m_W^4} \quad (11.67)$$

Neutrinos



Neutrinos

Because of Eqs. (11.59) and (11.62), the temperature dependence of the Hubble constant is

$$H(T) = 1.66 g_*^{1/2} \frac{T^2}{m_P} \quad (11.68)$$

where m_P is the Planck mass, $m_P c^2 = 1.22 \times 10^{19}$ GeV (see later, Eq. 12.25).

Neutrino equilibrium possible as long as $\Gamma_{\text{weak}} > H$, i.e., (inserting exact numbers)

$$k_B T_{\text{dec}} \gtrsim \left(\frac{500 c^6 m_W^4}{m_P} \right)^{1/3} \sim 1 \text{ MeV} \quad (11.69)$$

Neutrinos decouple ~ 1 s after the big bang.

This follows from Eq. (11.63), remembering that for this phase, $g_* \sim 10$.

Since decoupling, primordial neutrinos just follow expansion of universe, virtually no interaction with “us” anymore.

Neutrinos



Entropy

The entropy of particles is defined through

$$S = \frac{E + PV}{T} \quad (11.70)$$

Important for cosmology: relativistic limit. Define the entropy density,

$$s = \frac{S}{V} = \frac{E/V + P}{T} = \frac{u + P}{T} \approx \frac{4u}{3T} \quad (11.71)$$

(last step for relativistic limit; Eq. 11.40)

Inserting Eq. (11.39) ($u \propto (7/8)T^4$, $7/8$ for Fermions only) gives

$$s = \frac{7}{8} \frac{2\pi^2}{45} g k_B \left(\frac{k_B T}{\hbar c} \right)^3 = \frac{7}{8} \frac{2\pi^4}{45 \zeta(3)} k_B n \quad (11.72)$$

Since $s \propto n$ for backgrounds, the parameter $\eta = n_{\text{CMBR}}/n_{\text{baryons}}$ is often called the "entropy per baryon".

Neutrinos



For a mixture of backgrounds, Eq. (11.72) gives

$$s = g_{*,S} \cdot \frac{2\pi^2}{45} \left(\frac{k_B T}{\hbar c} \right)^3 \quad (11.73)$$

where $g_{*,S}$ is the analogue to g_* (Eq. 11.55),

$$g_{*,S} = \sum_{\text{bosons}} g_B \left(\frac{T_B}{T} \right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_F \left(\frac{T_F}{T} \right)^3 \quad (11.74)$$

Note that if the species are not at the same temperature, $g_* \neq g_{*,S}$.

Entropy per mass today:

$$\frac{S}{M} = \frac{10^{16}}{\Omega h^2} \text{erg K}^{-1} \text{g}^{-1} \quad (11.75)$$

while the entropy gain of heating water at 300K by 1K is $\sim 1.4 \times 10^5 \text{erg K}^{-1} \text{g}^{-1}$.

\Rightarrow "Human attempts to obey 2nd law ... are swamped by ... microwave background" (Peacock, 1999, p. 277).

$\Rightarrow S = \text{const.}$ for universe to very good approximation.

\Rightarrow Universe expansion is adiabatic!

Neutrinos



Reheating

After decoupling of neutrinos, neutrino distribution just gets redshifted (similar to CMBR, Eq. 11.19):

$$\frac{T_\nu}{T_{\text{dec}}} = \frac{R_{\text{dec}}}{R(t)} \Rightarrow T_\nu \propto R^{-1} \quad (11.76)$$

On the other hand, the temperature of the universe is

$$T \propto g_{*,S}^{-1/3} R^{-1} \quad (11.77)$$

This follows from $S/V \propto T^3$ (Eq. 11.73), $V \propto R^3$, and $S = \text{const.}$ (adiabatic expansion of the universe).

\Rightarrow as long as $g_{*,S} = \text{const.}$ we have $T_\nu = T$

\Rightarrow Immediately after decoupling, neutrino background appears as if it is still in equilibrium.

However: Temperature at neutrino decoupling is $\sim 2m_e c^2$. But, for $kT_{\text{BB}} < 2m_e c^2$, pair creation,

$$\gamma + \gamma \leftrightarrow e^- + e^+ \quad (11.78)$$

is kinematically impossible.

\Rightarrow Shortly after neutrino decoupling: e^\pm annihilation

$\Rightarrow g_{*,S}$ changes!

\Rightarrow We expect that $T_{\text{CMBR}} \neq T_\nu$.

Neutrinos



Reheating

Difference in $g_{*,S}$:

• before annihilation: $e^-, e^+, \gamma \Rightarrow g_{*,S} = 2 + 2 \cdot 2 \cdot (7/8) = 11/2$.

• after annihilation: $\gamma \Rightarrow g_{*,S} = 2$

But: the total entropy for particles in equilibrium is conserved ("expansion is adiabatic"):

$$g_{*,S}(T_{\text{before}}) \cdot T_{\text{before}}^3 = g_{*,S}(T_{\text{after}}) \cdot T_{\text{after}}^3 \quad (11.79)$$

such that

$$T_{\text{after}} = \left(\frac{11}{4} \right)^{1/3} T_{\text{before}} \sim 1.4 \cdot T_{\text{before}} \quad (11.80)$$

Since $T_{\text{after}} > T_{\text{before}}$: "reheating".

Note that in reality the annihilation is not instantaneous and T decreases (albeit less rapidly) during "reheating"...

\Rightarrow Since neutrino-background does not "see" annihilation it just continues to cool

\Rightarrow current temperature of neutrinos is

$$T_\nu = \left(\frac{4}{11} \right)^{1/3} T_{\text{CMBR}} \sim 1.95 K \quad (11.81)$$

Neutrinos



History

After reheating: universe consists of p, n, γ (and e^- to preserve charge neutrality)

⇒ Ingredients for Big Bang Nucleosynthesis (BBN):

Historical perspective: Cross section to make Deuterium:

$$\langle\sigma v\rangle(p+n \rightarrow D+\gamma) \sim 5 \times 10^{-20} \text{ cm}^3 \text{ s}^{-1} \quad (11.82)$$

Furthermore, we need temperatures of $T_{\text{BBN}} \sim 100 \text{ keV}$, i.e., $t_{\text{BBN}} \sim 200 \text{ s}$ (Eq. 11.63).

By Eq. (11.20) this implies a particle density of

$$n \sim \frac{1}{\langle\sigma v\rangle \cdot t_{\text{BBN}}} \sim 10^{17} \text{ cm}^{-3} \quad (11.83)$$

Today: Baryon density $n_B \sim 10^{-7} \text{ cm}^{-3}$. Since $n \propto R^{-3}$,

$$T(\text{today}) = \left(\frac{n_B}{n}\right)^{1/3} \cdot T_{\text{BBN}} \sim 10 \text{ K} \quad (11.84)$$

pretty close to the truth...

The above discussion was first asserted by George Gamov and coworkers in 1948, and was the first prediction of the cosmic microwave background radiation!

Observations: BBN is **required** by observations, since no other production region for Deuterium known, and since He-abundance $\sim 25\%$ by mass everywhere.

Big Bang Nucleosynthesis: Theory

1



Proton/Neutron

Initial conditions for BBN: Set by Proton-Neutron-Ratio.

For $t \ll 1 \text{ s}$, equilibrium via weak interactions:

$$\begin{aligned} n &\longleftrightarrow p + e^- + \bar{\nu}_e \\ \nu_e + n &\longleftrightarrow p + e^- \\ e^+ + n &\longleftrightarrow p + \bar{\nu}_e \end{aligned} \quad (11.85)$$

Reactions fast as long as particles relativistic.

But once $T \sim 1 \text{ MeV}$: n, p become non-relativistic

⇒ Boltzmann statistics applies (or use Eq. 11.51):

$$\frac{n_n}{n_p} = e^{-\Delta mc^2/k_B T} = e^{-1.3 \text{ MeV}/k_B T} \quad (11.86)$$

⇒ Suppression of n with respect to p because of larger mass ($m_n c^2 = 939.57 \text{ MeV}$, $m_p c^2 = 938.27 \text{ MeV}$)

Big Bang Nucleosynthesis: Theory

2



Proton/Neutron

As usual, the n, p abundance freezes out when $\Gamma \gg H$.

For the neutron, proton equilibrium, the reaction rate is

$$\Gamma(\nu_e + n \leftrightarrow p + e^-) \sim 2.1 \left(\frac{T}{1 \text{ MeV}}\right)^5 \text{ s}^{-1} \quad (11.87)$$

The neutron abundance freezes out at $k_B T \sim 0.8 \text{ MeV}$ ($t = 1.7 \text{ s}$), such that $n_n/n_p = 0.2$

After that: Neutron decay ($\tau_n = 886.7 \pm 1.2 \text{ s}$).

⇒ Nucleosynthesis has to be over before neutrons are decayed away!

⇒ Nucleosynthesis only takes a few minutes at most!

Big Bang Nucleosynthesis: Theory

3



Deuterium

The first step in nucleosynthesis is the formation of deuterium (binding energy $E_B = 2.225 \text{ MeV}$, i.e., $1.7(m_n - m_p)c^2$):



Note: Both **fusion** and photodisintegration are possible:

$$\Gamma_{\text{fusion}} = n_B \langle\sigma v\rangle \quad (11.89)$$

$$\Gamma_{\text{photo}} = n_\gamma \langle\sigma v\rangle e^{-E_B/k_B T} \quad (11.90)$$

At first: photodisintegration dominates since $\eta^{-1} = n_\gamma/n_B \sim 10^{10}$ (see Eq. 11.72).

Build up of D is only possible once $\Gamma_{\text{fusion}} > \Gamma_{\text{photo}}$, i.e., when

$$\frac{n_\gamma}{n_B} e^{-E_B/k_B T} \sim 1 \quad (11.91)$$

Inserting numbers shows that

Deuterium production starts at $k_B T \sim 100 \text{ keV}$, or $t \sim 100 \text{ s}$.

Big Bang Nucleosynthesis: Theory

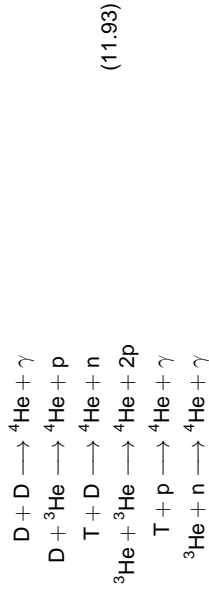
4

**Heavier Elements**

Once deuterium present:
nucleosynthesis of lighter elements:



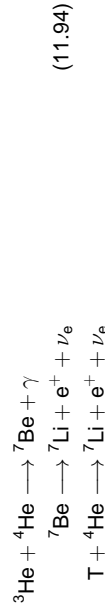
production of ${}^4\text{He}$:



Big Bang Nucleosynthesis: Theory

**Heavier Elements**

Element gap at $A = 5$ can be overcome to produce Lithium:



Gap at $A = 8$ prohibits production of heavier isotopes.

\implies Major product of BBN: ${}^4\text{He}$.

Mass fraction of ${}^4\text{He}$ can be estimated assuming all neutrons incorporated into ${}^4\text{He}$

\implies number density of H = number of remaining protons, i.e., mass fraction

$$X = \frac{n_p - n_n}{n_p + n_n} \quad (11.95)$$

and

$$Y = 1 - \frac{n_p - n_n}{n_p + n_n} = 2 \left(1 + \frac{n_p}{n_n} \right)^{-1} \quad (11.96)$$

Because of neutron decay, at $k_B T = 0.8 \text{ MeV}$: $n_n/n_p = 1/7$, such that

BBN predicts primordial He-abundance of $Y = 0.25$.

Big Bang Nucleosynthesis: Theory

**Remarkable Things**

Note the following coincidences:

1. Freeze out of nucleons simultaneous to freeze out of neutrinos.
2. . . . and parallel to electron-positron annihilation.
3. Expansion is slow enough that neutrons can be bound to nuclei.
 \implies Long chain of coincidences makes our current universe possible!

Big Bang Nucleosynthesis: Theory

**Detailed Calculations**

1. Generally, BBN operates as a function of the entropy per baryon, η . Remember that the entropy density for a baryon is

$$s = \frac{7}{8} \frac{2\pi^2}{45} g k_B \left(\frac{k_B T}{\hbar c} \right)^3 = \frac{7}{8} \frac{2\pi^4}{45 \zeta(3)} k_B n \quad (11.72)$$

and therefore the entropy per baryon is

$$\eta = \frac{n_{\text{CMBR}}}{n_{\text{baryons}}} \quad (11.97)$$

Note that η is related to Ω in baryons, Ω_B :

$$\Omega_B = 3.67 \times 10^7 \cdot \eta \quad (11.98)$$

(since η, Ω determine expansion behavior)

\implies Perform computations as function of η !

2. Since Y is set by n_p/n_n
 \implies He abundance is relatively independent from η

Big Bang Nucleosynthesis: Theory



Confrontation with WMAP

As we will see later: fluctuations in cosmic microwave background allow for a tight determination of cosmological parameters.

Best results so far from Wilkinson Microwave Anisotropy Probe (WMAP); see Spergel et al. 2007):

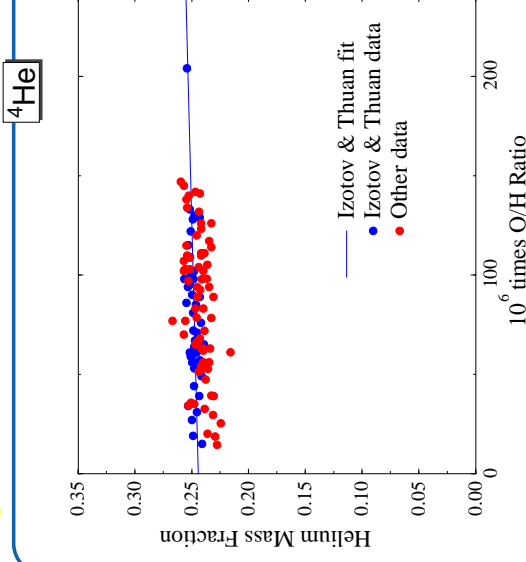
$$\Omega_b h^2 = 0.02233^{+0.00072}_{-0.00091} \quad (11.99)$$

With the most modern BBN calculations (Kneller & Steigman, 2004), this gives (Molaro, 2007):

Element	SBBN+WMAP
Y_p	$0.2482^{+0.0004}_{-0.0003}$
$^3\text{He}/\text{H}$	$(10.5 \pm 0.6) \times 10^{-6}$
D/H	$(25.7^{+1.7}_{-1.3}) \times 10^{-6}$
Li/H	$(4.41^{+0.3}_{-0.4}) \times 10^{-10}$

⇒ Can use WMAP parameters and BBN theory to compare BBN theory with measurements

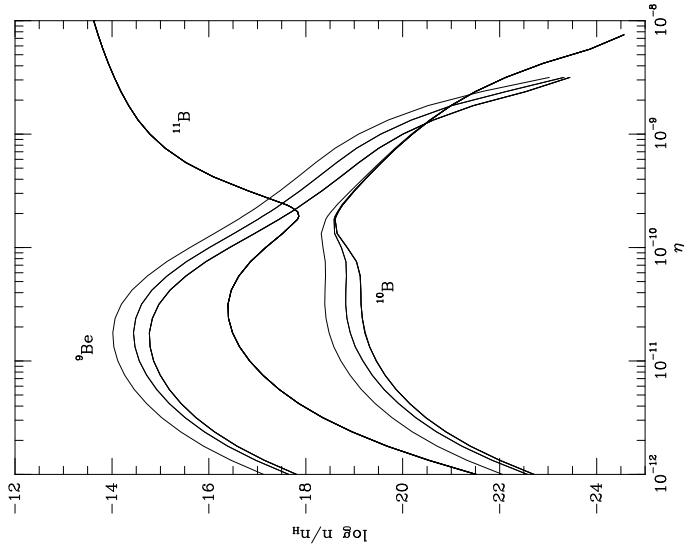
Big Bang Nucleosynthesis: Theory



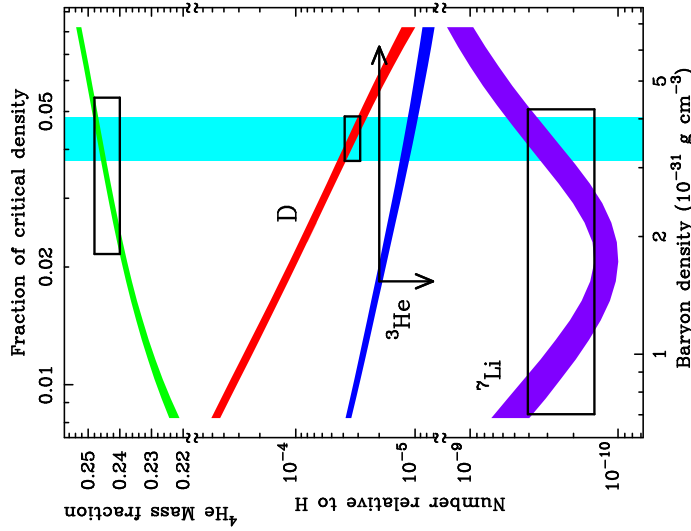
^4He produced in stars
 ⇒ extrapolate to zero metallicity in systems of low metallicity (i.e., minimize stellar processing).
 Best determination from He II → He I recombination lines in H II regions (metallicity ~ 20% solar).
 Result: Linear correlation He vs. O
 ⇒ extrapolate to zero oxygen to obtain primordial abundances.
 Result: $Y = 0.234 \pm 0.005$ (Olive, 1999).

(Burles et al., 1999, Fig. 4)

Nucleosynthesis: Observations



Intermediate mass abundances as function of η (Olive, 1999, Fig. 5)

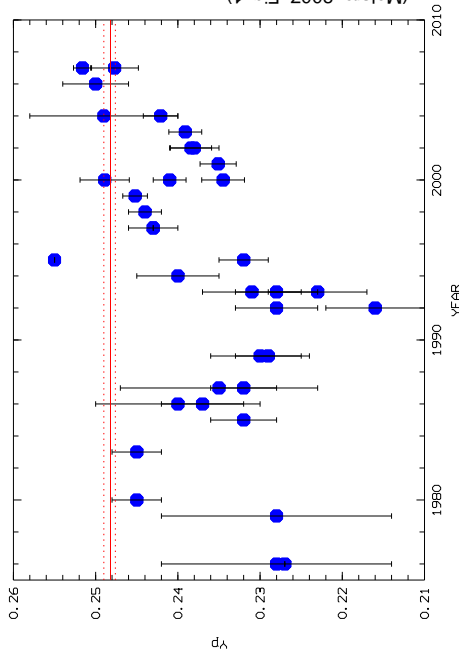


BBN observations strongly constrain Ω_{Baryons} .

(Burles et al., 1999, Fig. 1)



WMAP BBN and He



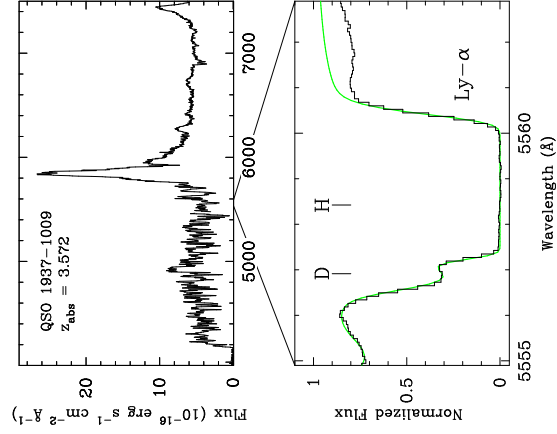
After improving He recombination physics and intrinsic absorption, He abundances are now in agreement with BBN prediction using Ω_B from WMAP.

Nucleosynthesis: Observations

2



Deuterium



Stars destroy D in fusion processes

⇒ use as non-processed material as possible!

Ly α forest: absorption of quasar light by intervening material

⇒ Some absorption lines in the Ly α forest show asymmetric line structure caused by primordial deuterium.

Remember the Balmer formula:

$$\frac{1}{\lambda_{n,m}} = R_H \left(\frac{1}{m} - \frac{1}{n} \right) \quad (11.100)$$

with Rydberg constant

$$R_H = \frac{m_e m_p}{m_e + m_p} \frac{c^4}{8\pi \epsilon_0^2 h^3} \quad (11.101)$$

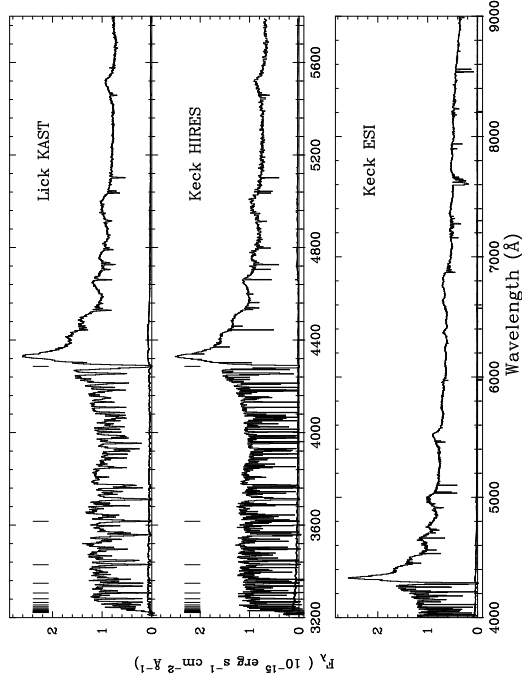
(QSO 1937-1009; top: 3 m Lick, bottom: Keck; Burles et al., 1999, Fig. 2)

Nucleosynthesis: Observations

3



Deuterium



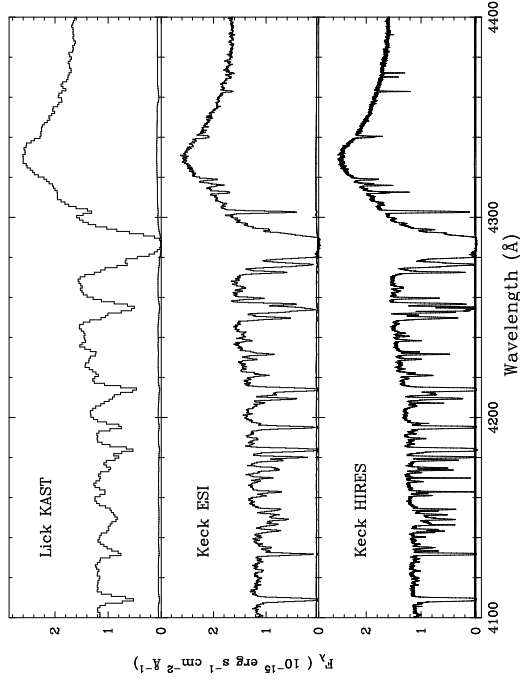
(Kirkman et al., 2003, Fig. 1): Lyman forest against three QSOs

Nucleosynthesis: Observations

4



Deuterium



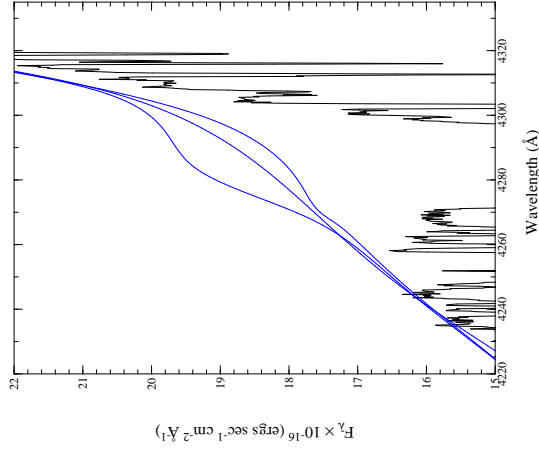
(Kirkman et al., 2003, Fig. 2): use absorption close to 4285 Å to measure D/H

Nucleosynthesis: Observations

5



Deuterium



To measure abundances, measure column from the optical depth:

$$\tau(\lambda) = n\sigma(\lambda)\ell = N\sigma(\lambda) \quad (11.102)$$

where σ : absorption cross section of line, N : column density. This can be measured from

$$I_{\text{obs}}(\lambda) = I_{\text{cont}}(\lambda)e^{-\tau(\lambda)} \quad (11.103)$$

⇒ Need to know the continuum, I_{cont}

Very difficult to do in Ly α forest (see Figure)

Currently best result for D/H (Kirkman et al., 2003):

$$D/H = 2.78^{+0.44}_{-0.38} \times 10^{-5}$$

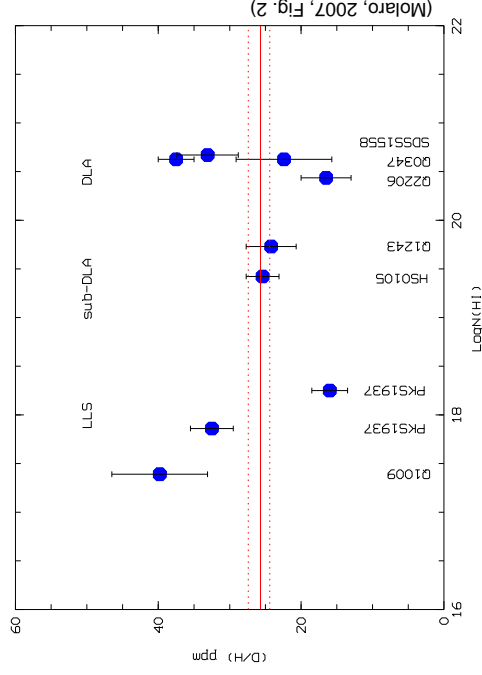
Corresponding to $\eta = 5.9 \pm 0.5 \times 10^{-10}$ or $\Omega_B h^2 = 0.0214 (\pm 9.3\%)$.

Nucleosynthesis: Observations

6



WMAP BBN and D



Measured deuterium abundances agree with WMAP predictions

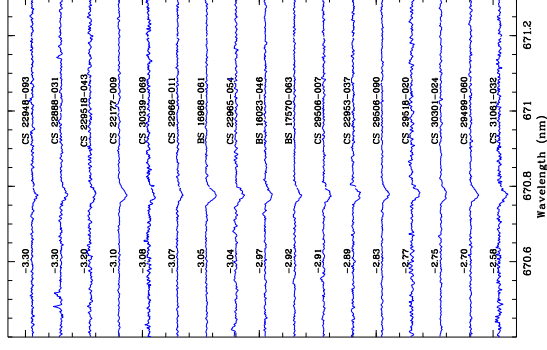
Although there are issues with Milky Way deuterium abundances...

Nucleosynthesis: Observations

7



Lithium



Lithium lines (Li doublet at 6707 Å) are visible in some stars
⇒ allow measurement of Li abundance

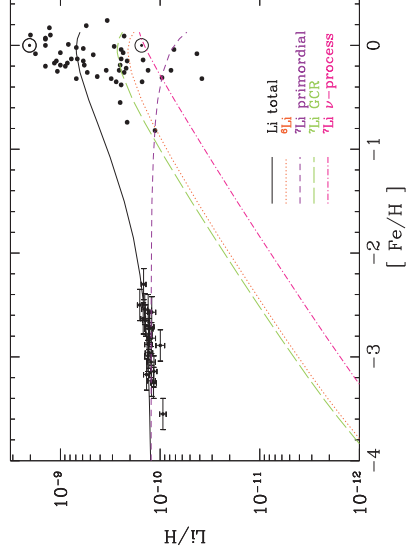
Li line as a function of [Fe/H] (Bonifacio et al., 2007, Fig. 1)

Nucleosynthesis: Observations

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Lithium

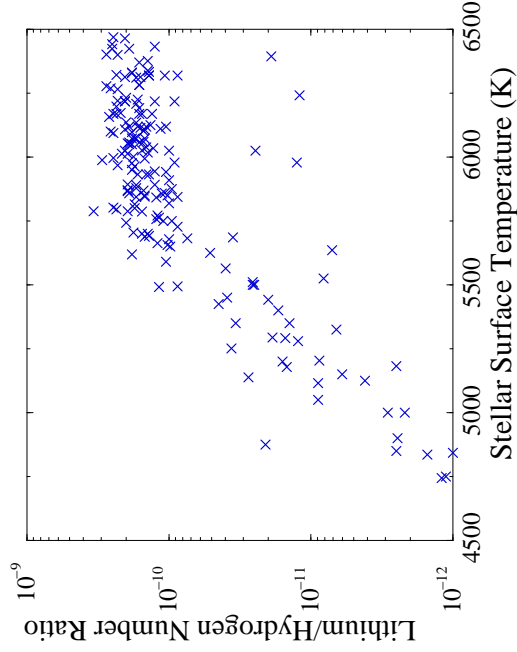


Olive (2005, astro-ph/0503065, Fig. 21)

To measure primordial Li abundance, one needs to go outside of the Galaxy (i.e., into a low metallicity environment), since inside the Galaxy, Galactic Cosmic Rays produce Li through spallation

Nucleosynthesis: Observations

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**Lithium**

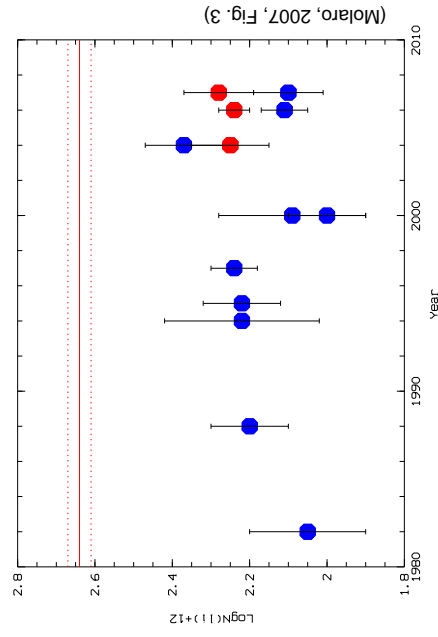
Spite & Spite (1982): Old halo stars with very low $[Fe/H]$ show primordial Lithium abundance, $7Li/H = 1.6 \times 10^{-10}$ "Spite plateau"

Lower temperature stars: outer convection zone \Rightarrow Li burning destroys Li.

(Burles et al., 1999, Fig. 5)

Nucleosynthesis: Observations

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**WMAP BBN and Li**

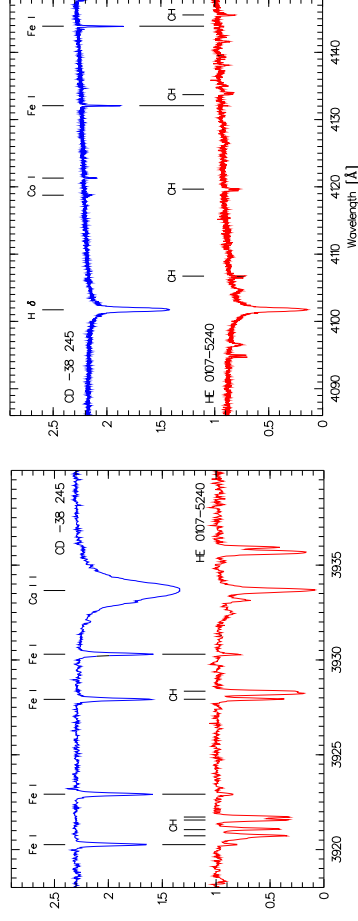
(Molaro, 2007, Fig. 3)

Lithium has a big problem!

Temperature sensitivity might have been underestimated, also rotational mixing, diffusion, and differences between 1D- and 3D-radiative transfer in stellar atmosphere models might play a role. However, no convincing solution has been proposed as of today.

Nucleosynthesis: Observations

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**Outlook: Population III**

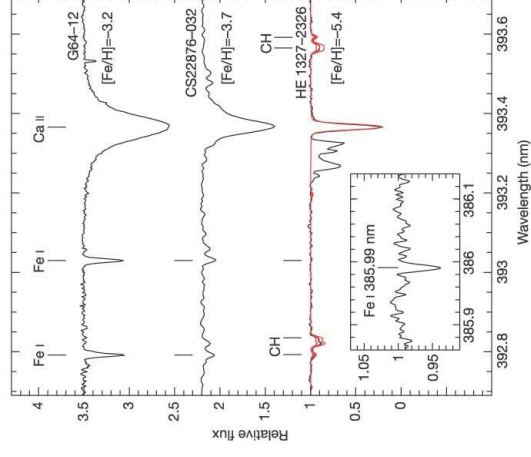
(HE0107-5240, metallicity 1/200000 solar; after Christlieb et al., 2002, Fig. 1)

Earliest stars should only have H, He, i.e., $Z = 0 \Rightarrow$ detection of such stars would enable the *direct* measure of primordial abundances.

"population III star", formed either from primordial gas cloud (and got some elements later through accretion from ISM), or from debris from type II SN explosion.

Nucleosynthesis: Observations

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**Outlook: Population III**

(Frebel et al., 2005, Fig. 2)

Lowest metallicity known:
HE 1327-2326, with Fe-abundance of 1/250000 solar

(Frebel et al., 2005, Fig. 1)

Nucleosynthesis: Observations

13

**Summary**

Summary: History of the universe after its first 0.01 s (after Islam, 1992, Ch. 7, see also Weinberg, The first three minutes).

$$t = 0.01 \text{ s} \quad T = 10^{11} \text{ K} \quad \rho \sim 4 \times 10^{11} \text{ g cm}^{-3}$$

Main constituents: γ , ν , $\bar{\nu}$, e^- - e^+ pairs.

No nuclei (unstable). n and p in thermal balance.

$$t = 0.1 \text{ s} \quad T = 3 \times 10^{10} \text{ K} \quad \rho \sim 3 \times 10^7 \text{ g cm}^{-3}$$

Main constituents: γ , ν , $\bar{\nu}$, e^- - e^+ pairs. No nuclei.

$n + \nu \leftrightarrow p + e^-$: mass difference becomes important, 40% n, 60% p (by mass).

Summary: Classical Big Bang

1

**Summary**

$$t = 1.1 \text{ s} \quad T = 10^{10} \text{ K} \quad \rho \sim 10^5 \text{ g cm}^{-3}$$

Neutrinos decouple, e^- - e^+ pairs start to annihilate. No nuclei.

25% n, 75% p

$$t = 13 \text{ s} \quad T = 3 \times 10^9 \text{ K} \quad \rho \sim 10^5 \text{ g cm}^{-3}$$

Reheating of photons, pairs annihilate, ν fully decoupled, deuterium still cannot form.

17% n, 83% p

$$t = 3 \text{ min} \quad T = 10^9 \text{ K} \quad \rho \sim 10^5 \text{ g cm}^{-3}$$

Pairs are gone, neutron decay becomes important, start of nucleosynthesis

14% n, 86% p

Summary: Classical Big Bang

2

**Summary**

$$t = 35 \text{ min} \quad T = 3 \times 10^8 \text{ K} \quad \rho \sim 0.1 \text{ g cm}^{-3}$$

game over

Next important event: $t \sim 300000$ years: Interaction CMB/matter stops ("last scattering", recombination).

Before we look at this, we look at the first 0.01 s: the very early universe

Summary: Classical Big Bang

3

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