Einführung in die Astronomie I

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Wintersemester 2006/2007

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Introduction

Astronomie an der FAU

NF im Vordiplom/BA: Gebraucht werden:

- Scheine Astronomie I und II
- Astronomisches Praktikum (Schein)

Astronomie im Hauptstudium Physik: hängt davon ab...

PWB: 10 SWS weiterführende Vorlesungen Astro-/Teilchenphysik, davon 2 SWS Theorie

nichtphysikalisches Wahlfach: wie NF im Hauptstudium, nur wenn Astronomie nicht im Vordiplom!

Nebenfächler (NF im Hauptstudium für Nichtphysiker):

- Astronomie I und II
- Eine weiterführende Vorlesung (2 SWS)
- Ein physikalischer Praktikumsschein (z.B. Astronomisches Praktikum)

Frühstudium: freiwillig,

- möglich sind Scheine Astronomie I und II

Scheine?!?

Bologna-Prozess ändert alles: politisch beschlossene Einführung von BA/MA Studiengängen.

Idee: Kumulative Abschlüsse, keine Prüfung am Ende.

An der FAU: Ab WS 2007/2008 BA, aber: Umstieg diesjähriger Erstsemester ist zum WS 2007/2008 (und nur dann!) erlaubt!

$\Longrightarrow$ Impliziert Notengebung in Vorlesungen ab WS 2006/2007!

$\Longrightarrow$ KLAUSUR am 30. Januar 2007

Klausur wird zu einem *benoteten* Schein führen, auf Wunsch sind bei Bestehen der Klausur auch *unbenotete* Scheine möglich.

Praktikum

Praktikum wird an der Dr. Karl Remeis-Sternwarte, Bamberg, als Blockpraktikum durchgeführt:

- 12.02.–23.02.2007: NF im Hauptstudium, schon 13 zugel.
- 26.02.–09.03.2007: NF im Hauptstudium, Lehramt
- 10.09.–21.09.2007: NF im Vordiplom/BA, Lehramt
- 24.09.–05.10.2007: NF im Vordiplom/BA, Lehramt

$\Longrightarrow$ Jeweils 15 Plätze pro Termin, wir lassen $30 - 13 - x$ NF Hauptstudium und $40 + x$ NF im Vordiplom/Lehramt zu, wo $x \geq 0$.

Eintragung in der Pause, die *vorläufige* Zulassung erfolgt im Laufe dieser Woche und wird nächste Woche mitgeteilt.

Endgültige Voraussetzung für eine Teilnahme am Praktikum sind normalerweise zwei Scheine aus Astronomie I und II.

Ausnahmen: Nebenfächler im Hauptstudium u.ä.

Textbooks

CARROLL & OSTLIE, 1996, *Modern Astrophysics*, Reading: Addison-Wesley, €80 (softcover), 1400 pp.

Advanced level, expects good physics background.
Recommended if you want to specialize in astronomy.

ZEILIK & GREGORY, 1998, *Introductory Astronomy & Astrophysics*, 4th ed., Thomson Learning, €64, 600 pp.

Intermediate level, self contained, but sometimes chaotic order.

KUTNER, 2003, *Astronomy: A Physical Perspective*, Cambridge: Cambridge Univ. Press, €55, 580 pp.

Modern physics based textbook, easy to read. Recommended.

Textbooks

KARTUNNEN, KRÖGER & OJA, 2003, *Fundamental Astronomy*, Heidelberg: Springer, €64 (softcover), 500 pp.

Good general overview of astronomy.
Recommended, especially for exam preparation.

UNSÖLD & BASCHEK, 2004, *Der neue Kosmos. Einführung in die Astronomie und Astrophysik*, Berlin: Springer, €50, 577 pp.

Very good overview of stellar astronomy, weaker on extragalactic astronomy.
Good secondary reading.

DE PATER & LISSAUER, 2004, *Planetary Sciences*, Cambridge: Cambridge University Press, €93, 544 pp.

The textbook of planetary science.
Good secondary reading.

Contents

17 Oct	**no lecture**
24 Oct	Introduction, History of Astronomy
31 Oct	Planets: Overview, Dynamics
07 Nov	Planets: Planetary Atmospheres
14 Nov	Planets: Planetary Surfaces and Interiors
21 Nov	Planets: Transneptunians, Asteroids, Comets, Meteorites
28 Nov	Measurement Methods: Telescopes, Coordinates
05 Dec	Extrasolar Planets
12 Dec	Stars: Distances, Luminosity, Masses, HRD
19 Dec	Stars: Formation, Structure
09 Jan	Stars: Evolution
16 Jan	Stars: Evolution, continued
23 Jan	End Stages: White Dwarfs, Neutron Stars, Black Holes
30 Jan	*Exam!*
06 Feb	Follow-up on exam

History of Astronomy

Introduction

Together with theology, astronomy one of the oldest professions in the world.

So what?

Astronomical nomenclature is still strongly influenced by this tradition

$\Longrightarrow$ appreciation of history of astronomy is required for understanding even today's astronomy (many terms used are based on this history).

Babylon

Babylonian astronomy: Earliest astronomy with influence on us: ~360 d year

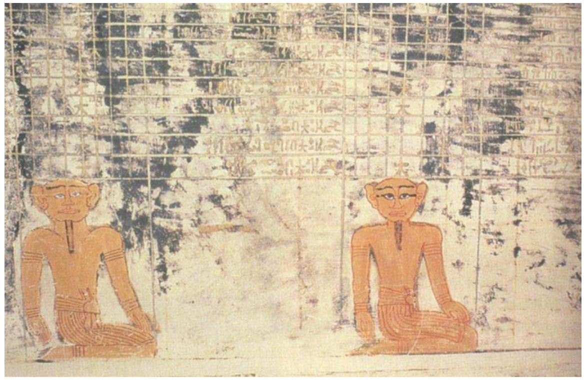
$\Longrightarrow$ sexagesimal system [360:60:60], 24h day, 12×30 d year,. . .

Enuma Elish myth (~1100BC): Universe is place of battle between Earth and Sky, born from world parents.

Note similar myth in the Genesis. . .

Image: Mul.Apin cuneiform tablet (British Museum, BM 86378, 8 cm high), describes rising and setting of constellations through the babylonian calendar. Summarizes astronomical knowledge as of before ~690 BC.

Egypt



Egyptian coffin lid showing two assistant astronomers, 2000. . . 1500 BC; hieroglyphs list stars ("decans") whose rise defines the start of each hour of the night.

(Aveni, 1993, p. 42)

~2000 BC: 365 d calendar (12×30 d plus 5 d extra), fixed to Nile flood (heliacal rising of Sirius), star clocks.

heliacal rising: first appearance of star in eastern sky at dawn, after it has been hidden by the Sun.

Greek/Roman, I

Atlas Farnese, 2c A.D., Museo Archeologico Nazionale, Napoli

Early Greek astronomy: folk tale astronomy (Hesiod (730?–? BC), *Works and Days*). Constellations.

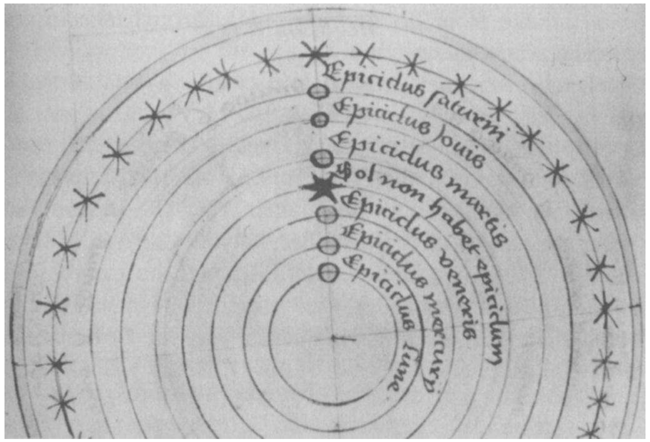
Thales (624–547 BC): Earth is flat, surrounded by water.

Anaxagoras (500–428 BC): Earth is flat, floats in nothingness, stars are far away, fixed on sphere rotating around us. Eclipses: due to Earth's shadow.

Eudoxus (408–355 BC): Geocentric, planets affixed to concentric crystalline spheres. First real model for planetary motions!

Aristarchus (310–230 BC): Determination of relative distance to Moon and Sun (factor 20).

Greek/Roman, II



Aristotle (384–322 BC, *de caelo*): Refinement of Eudoxus model: add spheres to ensure smooth motion

$\Longrightarrow$ Universe filled with crystalline spheres (*nature abhors vacuum*).

$\Longrightarrow$ Central philosophy until ∼1450AD!

Hipparchus (?? – ∼127 BC): Refinement of geocentric Aristotelian model into tool to make predictions.

Greek/Roman, III

(Aveni, 1993, p. 58)

Ptolemaeus (∼140AD): *Syntaxis* (aka Almagest): Refinement of Aristotelian theory into model useable for computations

$\Longrightarrow$ Ptolemaic System.

Renaissance, I

Nicolaus Copernicus (1473–1543): Earth centred Ptolemaic system is too complicated, a Sun-centred system is more elegant.

Renaissance, III

(Gingerich, 1993, p. 165)

Nicolaus Copernicus (1473–1543): Earth centred Ptolemaic system is too complicated, a Sun-centred system is more elegant:

De revolutionibus orbium coelestium: “In no other way do we perceive the clear harmonious linkage between the motions of the planets and the sizes of their orbs.”

Copernican principle: The Earth is not at the center of the universe.

Renaissance, IV

Tycho Brahe (1546–1601): Visual planetary positions of highest precision reveal flaws in Ptolemaic positions.

Renaissance, V

Johannes Kepler (1571–1630): Planets orbit on ellipses around Sun, not on circles, laws of motion.

Galileo Galilei (1564–1642): Moons of Jupiter, moving around Jupiter (Kepler $\Longrightarrow$ similar to heliocentric model!)...

Newton

(Newton, 1730)

Isaac Newton (1642–1727): Newton’s laws, physical cause for shape of orbits is gravitation (*De Philosophiae Naturalis Principia Mathematica*, 1687).

$\Longrightarrow$ Begin of modern physics based astronomy.

Aveni, A. F., 1993, Ancient Astronomers, (Washington, D.C.: Smithsonian Books)

Gingerich, O., 1993, The Eye of Heaven – Ptolemy, Copernicus, Kepler, (New York: American Institute of Physics)

Newton, I., 1730, Opticks, Vol. 4th, (London: William Innys), reprint: Dover Publications, 1952

The Planets: Overview

What is a planet?

First, need to look at the definition of a planet.

Historical background:

- antiquity–1781: 6 planets: Mercury, Venus, Earth, Mars, Jupiter, Saturn
- 1781: Wilhelm Herschel discovers Uranus $\Longrightarrow$ 7 planets
- 1801: Giuseppe Piazzi: Ceres $\Longrightarrow$ 8 planets
- 1802: Heinrich Olbers: Pallas $\Longrightarrow$ 9 planets, Herschel coins term “asteroid”.
- 1804: Karl Harding: Juno $\Longrightarrow$ 10 planets
- 1807: Heinrich Olbers: Vesta $\Longrightarrow$ 11 planets
- 1845: Karl Hencke: Astrea $\Longrightarrow$ 12 planets
- 1846: Johann Gottfried Galle discovers Neptune $\Longrightarrow$ 13 planets
- 1847: Karl Hencke: Hebe $\Longrightarrow$ 14 planets
- 1848: Andrew Graham: Metis $\Longrightarrow$ 15 planets
- 1849: Annibale de Gasparis: Hygiea $\Longrightarrow$ 16 planets
- 1849: Annibale de Gasparis: Parthenope $\Longrightarrow$ 17 planets
- Sometime in late 1800s: Asteroids are not planets $\Longrightarrow$ 8 planets

What is a planet?

- 1930: Clyde Tombaugh: Pluto ($d = 2400$ km) $\Longrightarrow$ 9 planets
- 2002: Chad Trujillo & Michael Brown 50000 Quaoar ($d \sim 1300$ km)
- 2003: Brown et al.: 90377 Sedna (1200 km $\lesssim d \lesssim$ 1800 km)
- 2005: Brown et al.: 2003 UB313 (aka “Xena”) ($d \sim 2400$ km)
  $\Longrightarrow$ 136199 Eris
  $\Longrightarrow$ 10 planets ?!?
- BUT: High frequency of discovering transneptunian objects

Summer 2006: International Astronomical Union General Assembly, Prague

$\Longrightarrow$ Resolution GA26/5 and 6:Definition of a planet

$\Longrightarrow$ 8 planets

RESOLUTION 5

Definition of a Planet in the Solar System

Contemporary observations are changing our understanding of planetary systems, and it is important that our nomenclature for objects reflect our current understanding. This applies, in particular, to the designation "planets". The word "planet" originally described "wanderers" that were known only as moving lights in the sky. Recent discoveries lead us to create a new definition, which we can make using currently available scientific information.

The IAU therefore resolves that planets and other bodies, except satellites, in our Solar System be defined into three distinct categories in the following way:

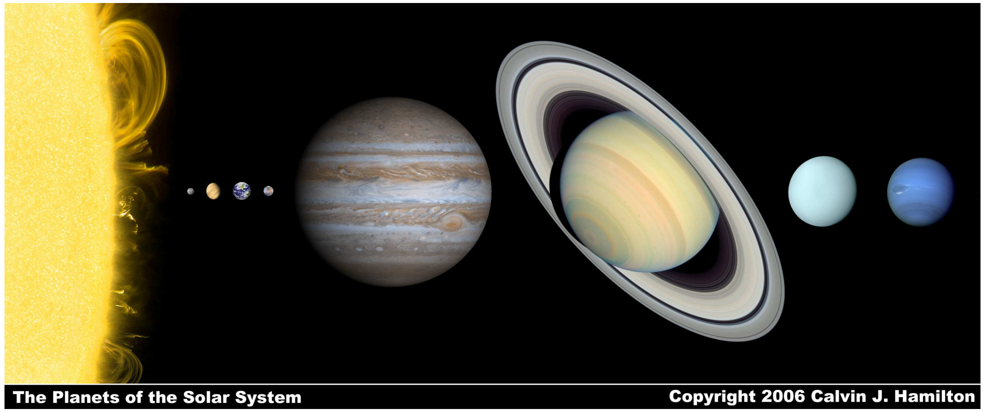
(1) A planet[1] is a celestial body that
- (a) is in orbit around the Sun,
- (b) has sufficient mass for its self-gravity to overcome rigid body forces so that it assumes a hydrostatic equilibrium (nearly round) shape, and
- (c) has cleared the neighbourhood around its orbit.

(2) A "dwarf planet" is a celestial body that
- (a) is in orbit around the Sun,
- (b) has sufficient mass for its self-gravity to overcome rigid body forces so that it assumes a hydrostatic equilibrium (nearly round) shape[2],
- (c) has not cleared the neighbourhood around its orbit, and
- (d) is not a satellite.

(3) All other objects[3], except satellites, orbiting the Sun shall be referred to collectively as "Small Solar System Bodies".

1 The eight planets are: Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune.
2. An IAU process will be established to assign borderline objects into either dwarf planet and other categories.
3. These currently include most of the Solar System asteroids, most Trans-Neptunian Objects (TNOs), comets, and other small bodies.

3–3



Relative sizes of the Sun and the planets

3–3

RESOLUTION 6

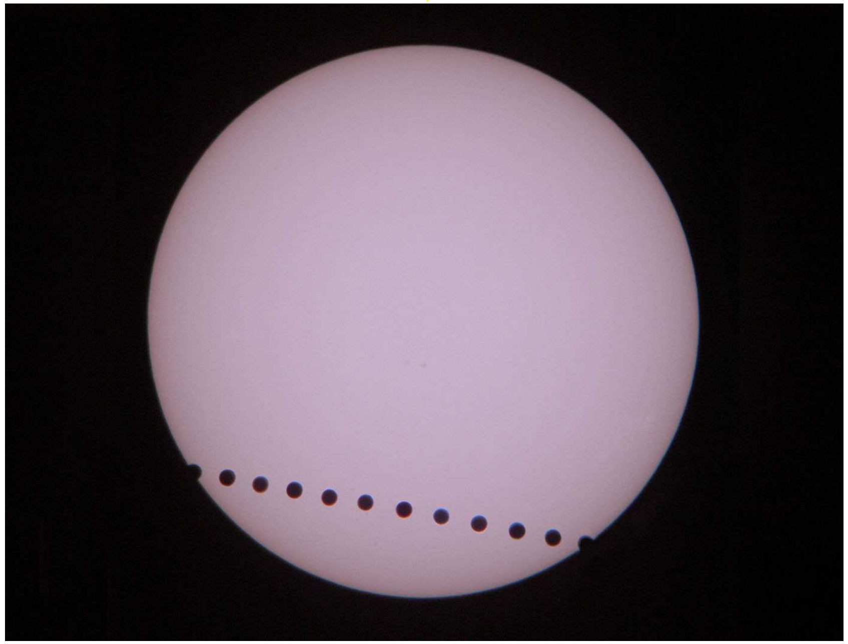
Pluto

The IAU further resolves:

Pluto is a "dwarf planet" by the above definition and is recognized as the prototype of a new category of Trans-Neptunian Objects[1].

1 An IAU process will be established to select a name for this category.

Venus Transit, 2004 June 8

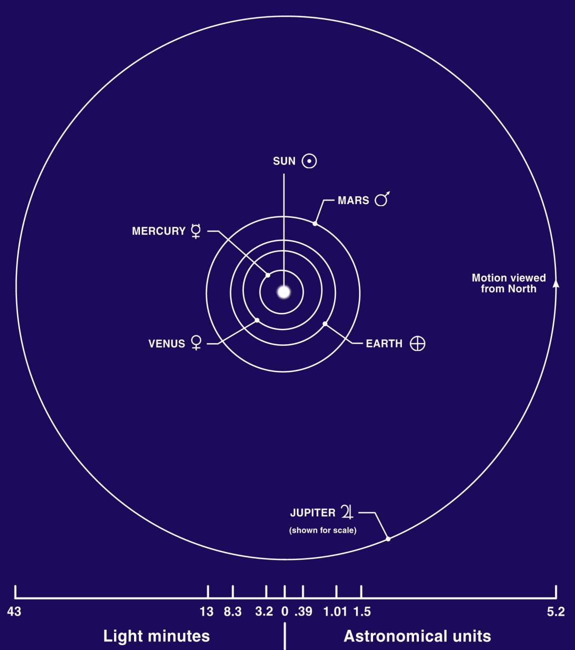


Elio Daniele, Palermo

Overview, I

Division of Solar System into two major types of planets:

1. Inner “Terrestrial” Planets: Mercury, Venus, Earth/Moon, Mars:
   ⟹ all similar to Earth (“rocks”).
   ⟹ *no moons* (Earth/Moon better called “twins”)



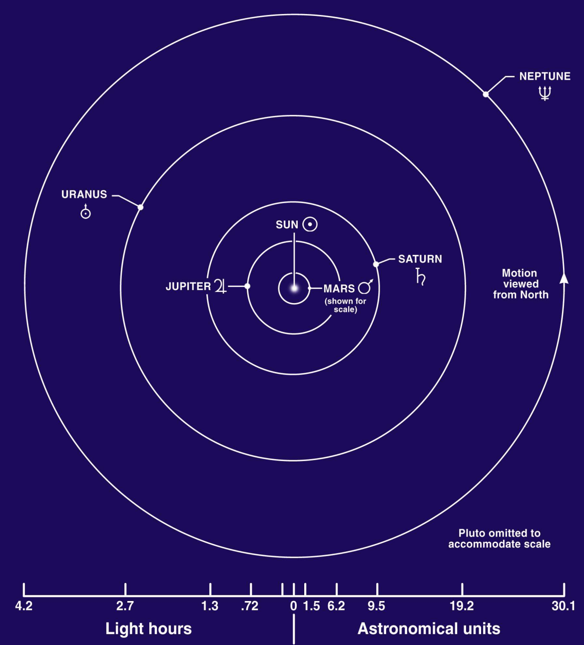
The Inner Planets (SSE, NASA)

Overview, III

Division of Solar System into two major types of planets:

1. Inner “Terrestrial” Planets: Mercury, Venus, Earth/Moon, Mars:
   ⟹ all similar to Earth (“rocks”).
   ⟹ *no moons* (Earth/Moon better called “twins”)
   ⟹ Moons of
2. Outer Planets: Jupiter, Saturn, Uranus, Neptune:
   ⟹ “gas giants”
   ⟹ all have *extensive moon systems*

Although not planets (i.e., motion not around Sun), large moons of gas giants are very similar in structure to terrestrial planets.

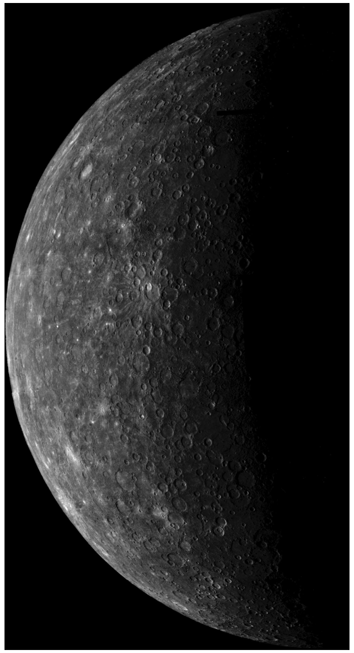


The Outer Planets (SSE, NASA)

Planets: Properties, I

		a [AU]	P_{orb} [yr]	i [°]	e	P_{rot}	$M/M_\oplus$	$R/R_\oplus$
Mercury	☿	0.387	0.241	7.00	0.205	58.8 d	0.055	0.383
Venus	♀	0.723	0.615	3.40	0.007	−243.0 d	0.815	0.949
Earth	⊕	1.000	1.000	0.00	0.017	23.9 h	1.000	1.00
Mars	♂	1.52	1.88	1.90	0.094	24.6 h	0.107	0.533
Jupiter	♃	5.20	11.9	1.30	0.049	9.9 h	318	11.2
Saturn	♄	9.58	29.4	2.50	0.057	10.7 h	95.2	9.45
Uranus	♅	19.2	83.7	0.78	0.046	−17.2 h	14.5	4.01
Neptune	♆	30.1	163.7	1.78	0.011	16.1 h	17.1	3.88
(Pluto	♇	39.2	248	17.2	0.244	6.39 d	0.002	0.19)

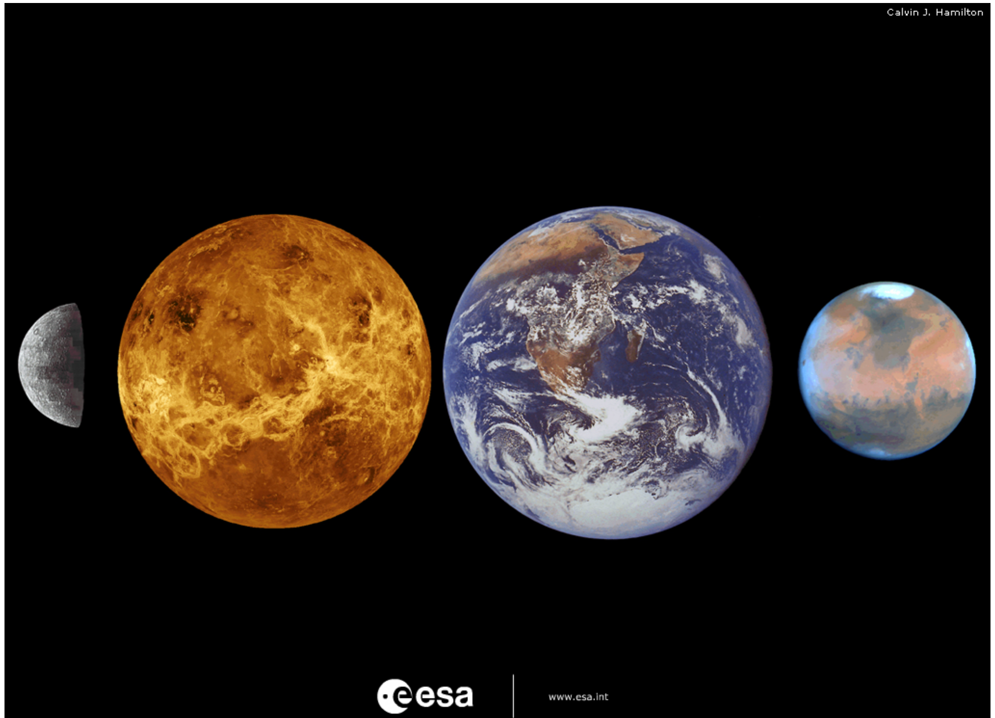
After Kutner, Appendix D;
a: semi-major axis P_{orb}: orbital period i: orbital inclination (wrt Earth's orbit)
e: eccentricity of the orbit P_{rot}: rotational period M: mass
R: equatorial radius
1 AU $= 1.496 \times 10^{11}$ m.

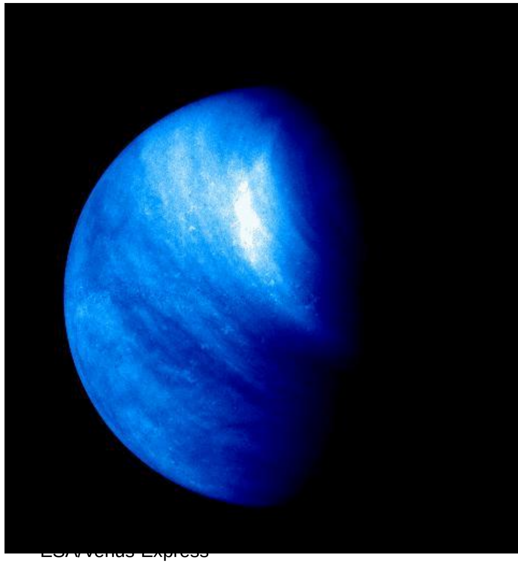


NASA/USGS

Mercury:

- not much larger than Moon
- densest of all terrestrial planets
- no evidence for atmosphere
- Rotation period: 59 d, 2/3 of orbital period.
- surface: impact craters and tectonics
- Only information available is from Mariner 10 (three flybys, 1974/1975)
- NASA mission "Messenger"
  (launched 2004 August 3, flyby 2008 and 2009, in orbit from 2011 on)
- ESA Mission Bepi Colombo, planned for ~ Aug. 2013, arrival Aug. 2019





ESA/Venus Express

Venus:

- similar size to Earth, similar structure
- insolation ~2× Earth
- very slow rotation (243 d, retrograde; $\Longrightarrow$ no B-field)
- very dense atmosphere: surface pressure ~90× Earth
- atmosphere: 96.5% CO_2, 3.5% N
  $\Longrightarrow$ strong greenhouse effect
  $\Longrightarrow$ surface temperature ~460°C.
- acid rain (yes, sulphuric acid!)

Information mainly from radar surveying from Earth and from Magellan (1990–1994), plus images from Pioneer Venus Probe (1979). Several landings (Venera, 1975/1981). Currently studied by ESA's Venus Express probe (launch April 2005, arrival April 2006).

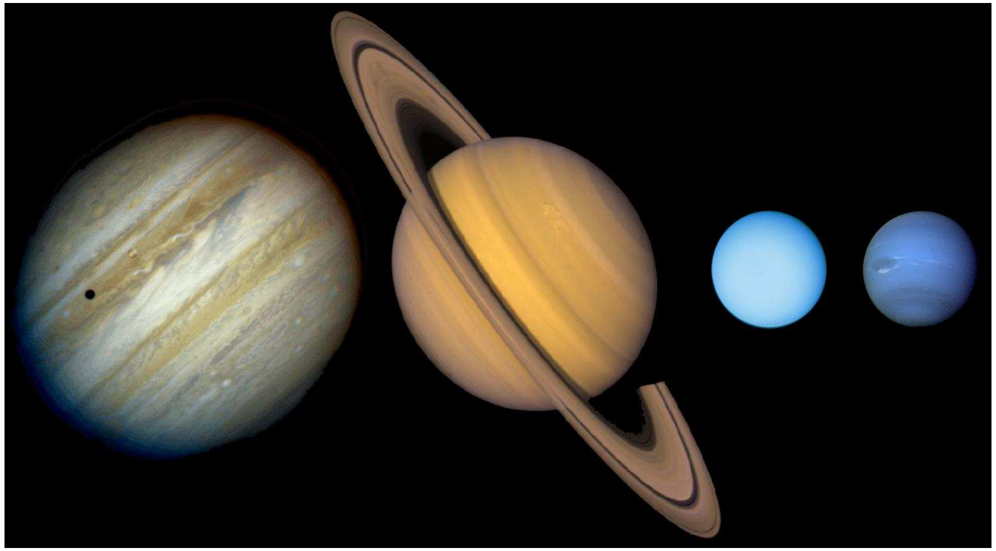
Earth/Moon, seen from Mars (NASA/Malin)

Earth:

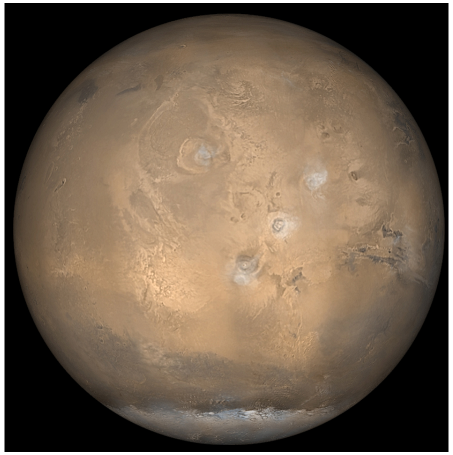
- double planet system
- Earth surface: *dominated by plate tectonics, erosion*
- atmosphere: 80% N_2, 20%O_2
  $\Longrightarrow$ moderate greenhouse effect
  $\Longrightarrow$ surface temperature $>0^{\circ}$C.
- water present

Moon:

- very similar to Mercury, overall
- Mariae (plains from massive impacts) and impact craters
- Rotation synchronous to orbit around Earth



The jovian planets, ©C.J. Hamilton

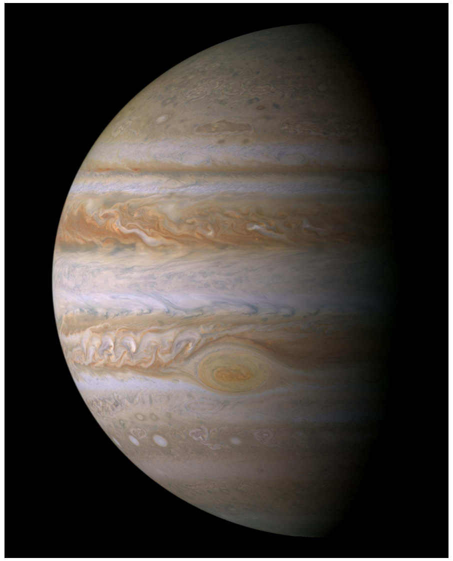


NASA, Mars Global Surveyor

Mars:

- smaller than Earth
- very low density ($\langle\rho\rangle \sim 3\,\mathrm{g\,cm^{-3}}$)
  $\Longrightarrow$ small core, probably Fe and Fe_xS_y,
- polar caps, seasons
- thin atmosphere, clouds, fog,...
- water sublimes
  $\Longrightarrow$ no liquid water today
- Volcanism (large shield volcanoes
  $\Longrightarrow$ no (?) plate tectonics)
- atmosphere: 95% CO_2
  $\Longrightarrow$ weak greenhouse effect
- two moons (captured asteroids)

Early Exploration through Mariner missions and Viking 1 and Viking 2 orbiters and landers in 1970s, recently, strong interest (NASA Mars Global Surveyor [MGS], ESA Mars Express, plus several landers). Currently best surveyed planet except for Earth.

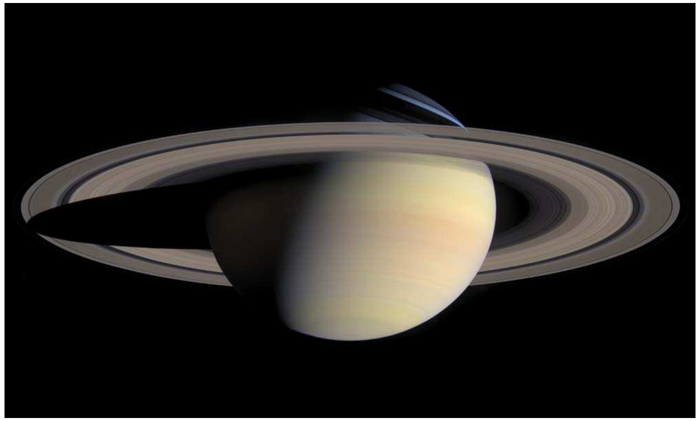


NASA/ESA, Cassini-Huyghens

Jupiter:

- Largest planet in solar system
- rapid rotation $\Longrightarrow$ severely flattened, banded atmosphere (Coriolis force), Great Red Spot
- strong magnetic field (strong radio emission)
- atmosphere: 75% H, 24% He (by mass), very close to solar
- differential rotation (rotation period 9h50m on equator, 9h55m on poles).
- strong magnetic field
- four major "Galilean" moons plus 59 small ones (as of Jan. 2005; all are captured asteroids)

Early Exploration 1970s through Pioneer 11 and 12, and then through the Voyager probes. Extensively studied by NASA's Galileo project (ended 2003 Sep 14).



NASA/ESA Cassini, 2004 Oct.

Saturn:

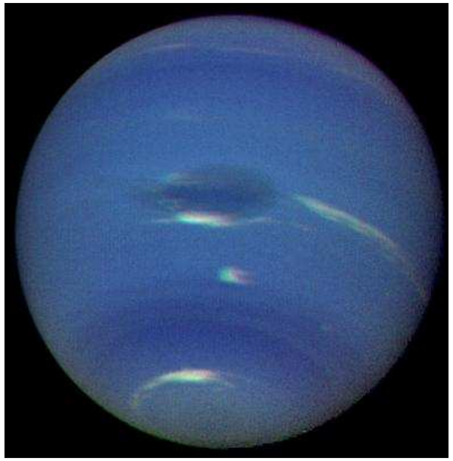
- similar to Jupiter, slightly smaller
- rapid rotation $\Longrightarrow$ flattened, banded atmosphere
- atmosphere: 75% H, 24% He (by mass), molecules etc. similar to Jupiter
- Rings!
- six major moons plus 27 small ones (as of Jan. 2005; mainly captured asteroids)
- Early Exploration in 1970s with Pioneer 11 and 12 and the Voyager probes.
  Studied since 2004 July 1 by NASA/ESA Cassini-Huygens project (duration: four years)

Uranus:

- atmosphere cold (59 K = -214°C)
  $\Longrightarrow$ ammonia has frozen out
- methane, hydrogen, and helium detected so far (less He than expected from Jupiter and Saturn!)
  $\Longrightarrow$ bluish color
- inclination of rotation axis: 98° ("rolling on ecliptic plane").
- small ring system
- five major moons in equatorial plane plus 22 small ones (as of Jan. 2004; captured asteroids)

NASA Voyager 2, 1986 Jan 10

Flyby of Voyager 2 in 1986 January, since then only remote sensing via Hubble Space Telescope (HST) and ground based instruments.



NASA Voyager 2

Neptune:

- atmosphere similar to Uranus, but more active; bright methane clouds above general cloud layer
- ring system (5 individual rings)
- Two major moons (Triton, 2720 km diameter(!) and Nereid 355 km), 11 captured asteroids

Flyby in 1989 August by Voyager 2, only HST since then (showed in 1995 that dark spot has vanished, detected new storm system)

Structure

Questions that we will deal with:

1. How do the planets move?
   Kepler's laws and their physical interpretation
2. What do planetary surfaces look like?
   craters, plate tectonics, volcanism
3. How do planetary atmospheres work?
   hydrostatic structure
4. What is the internal structure of the planets?
   hydrostatic structure (again)
5. Is the solar system normal?
   Are there planets elsewhere?

The Planets: Dynamics

Introduction

Johannes Kepler: Motion of planets governed by three laws:

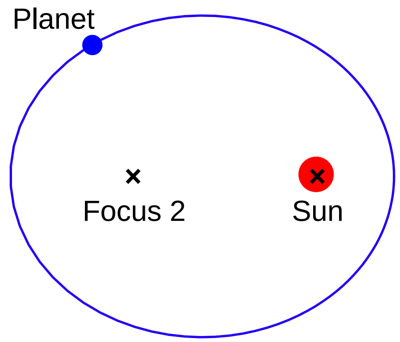
1. Each planet moves in an elliptical orbit, with the Sun at one focus of the ellipse. (“Astronomia Nova”, 1609)
2. A line from the Sun to a given planet sweeps out equal areas in equal times. (“Astronomia Nova”, 1609)
3. The square of the orbital periods of the planets is proportional to the cube of the major axes. (“Harmonice Mundi”, 1619)

Isaac Newton (“Principia”, 1687): Kepler’s laws are consequence of gravitational interaction between planets and the Sun, and the gravitational force is

$$\boldsymbol{F}_1 = -\frac{Gm_1m_2}{r_{12}^2}\frac{\boldsymbol{r}_{21}}{r_{12}} \quad (4.1)$$

where $\boldsymbol{F}_1$ is the gravitational force exerted on object 1, m_1, m_2 are the masses of the interacting objects, r their distance, and $\boldsymbol{r}_{21}/r_{12}$ the unit vector joining the objects, $\boldsymbol{r}_{21} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, $\boldsymbol{r}_{12} = -\boldsymbol{r}_{21}$ and $r_{12} = |\boldsymbol{r}_{12}| = |\boldsymbol{r}_{21}|$.

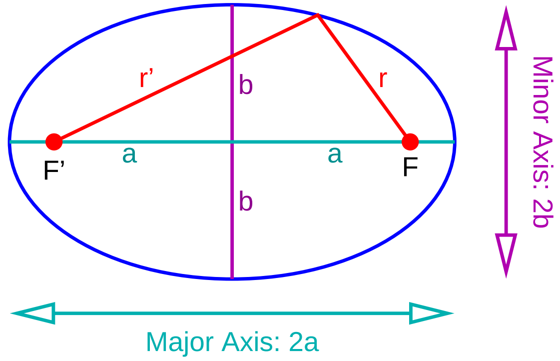
Keplers 1st Law



Kepler’s 1st Law: The orbits of the planets are ellipses and the Sun is at one focus of the ellipse.

For the planets of the solar system, the ellipses are almost circular, for comets they can be very eccentric.

Keplers 1st Law

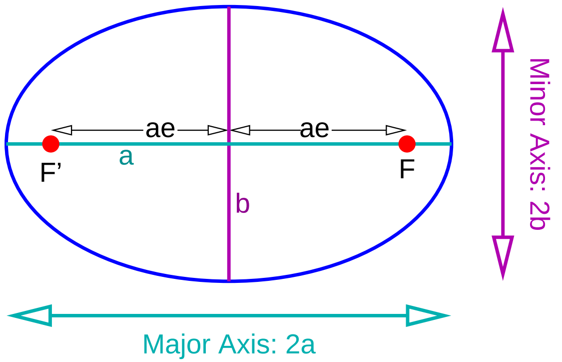


Definition: Ellipse = Sum of distances r, r' from any point on ellipse to two fixed points (foci, singular: focus), F, F', is constant:

$$r + r' = 2a \quad (4.2)$$

where a is called the semi-major axis of the ellipse.

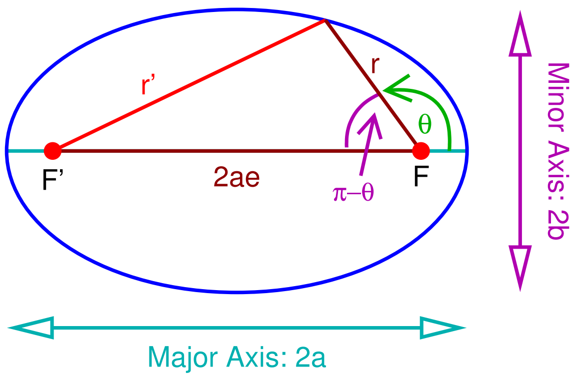
Keplers 1st Law



Definition: Eccentricity e: ratio between distance from centre of ellipse to focal point and semi-major axis.

So circles have $e = 0$.

Keplers 1st Law



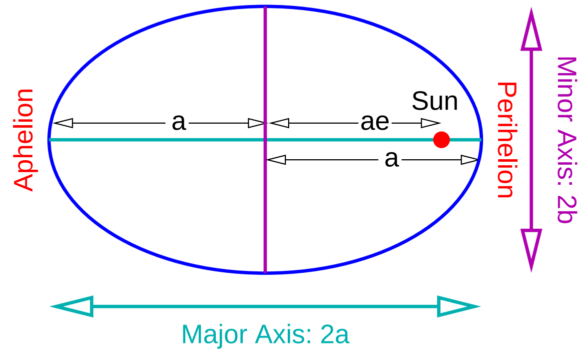
Law of cosines: $r'^2 = r^2 + (2ae)^2 - 2 \cdot r \cdot 2ae \cdot \cos(\pi - \theta)$

use $r + r' = 2a$ and solve for r to find the polar coordinate form of the ellipse:

$$r = \frac{a(1 - e^2)}{1 + e \cos\theta} \tag{4.3}$$

Check this for yourself! θ is called the *true anomaly*.

Keplers 1st Law

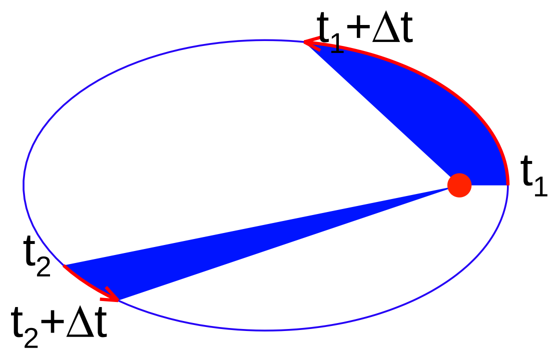


Finally, we need the closest and farthest point from a focus:

$$\begin{aligned} \text{closest point}: d_{\text{perihelion}} &= a - ae = a(1 - e) \\ \text{farthest point}: d_{\text{aphelion}} &= a + ae = a(1 + e) \end{aligned} \tag{4.4}$$

for stars: periastron and apastron,
for satellites circling the Earth: perigee and apogee.

2nd Law



Kepler's 2nd Law: The radius vector to a planet sweeps out equal areas in equal intervals of time.

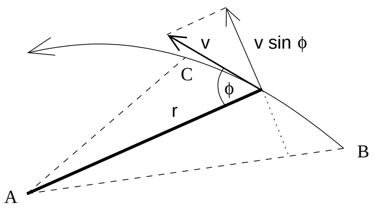
1. Kepler's 2nd Law is also called the *law of areas*.
2. perihelion: planet nearest to Sun $\Longrightarrow$ planet is fastest
3. aphelion: planet farthest from Sun $\Longrightarrow$ planet is slowest

Kepler's 2nd law is a direct consequence of the conservation of angular momentum. Remember that angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v} \tag{4.5}$$

and its absolute value is

$$L = mrv \sin\phi \tag{4.6}$$



To interpret the angular momentum, look at the figure at the left. The projection of the velocity vector perpendicular to the radius vector r is $v\sin\phi$. The distance traveled by the planet in an infinitesimally short time Δt is given by $\Delta x = \Delta t \cdot v\sin\phi$. Therefore, the area of the triangle ABC is given by

$$\Delta A = \frac{1}{2} r \Delta x = \frac{1}{2} r \Delta t v \sin\phi = \frac{L}{2m}\Delta t \tag{4.7}$$

Kepler's 2nd law states that the "sector velocity" $\mathrm{d}A/\mathrm{d}t$ is constant with time:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{L}{2m} = \text{const.} \tag{4.8}$$

To confirm that this claim is true, we need to prove that

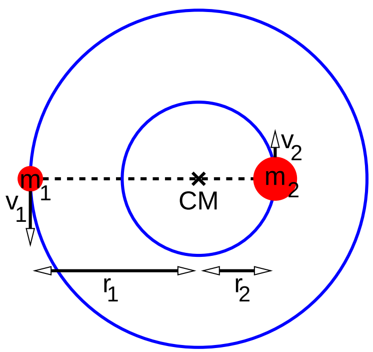
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2m}\frac{\mathrm{d}L}{\mathrm{d}t} = 0 \tag{4.9}$$

But $\mathrm{d}L/\mathrm{d}t$ is given by

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times \frac{GMm}{r^2}\frac{\boldsymbol{r}}{r} = 0 \tag{4.10}$$

since the cross product of a vector with itself is zero. Therefore, Kepler's 2nd law is true and is a consequence of the conservation of angular momentum for a central field.

Circular Motion



For an interpretation of Kepler's third law, consider the motion of two bodies with masses m_1 and m_2 on circular orbits with radii r_1 and r_2 around a point CM (see figure).

The reason for doing the computation with circular orbits is that the following discussion will be easier, however, all results from this section also apply to the general case of elliptical motion. This will be proven later in the lectures on Theoretical Mechanics.

The attractive force between the two points is given by Newton's law:

$$F_{\text{grav}} = G\frac{m_1 m_2}{R^2} = G\frac{m_1 m_2}{(r_1+r_2)^2} \tag{4.12}$$

In order to keep the two bodies on circular orbits, the gravitational force needs to be equal the centripetal force keeping each body on its circular orbit.

The centripetal force is

$$F_{\text{cent},1} = \frac{m_1 v_1^2}{r_1} = \frac{4\pi^2 m_1 r_1}{P^2}$$
$$F_{\text{cent},2} = \frac{m_2 v_2^2}{r_2} = \frac{4\pi^2 m_2 r_2}{P^2} \tag{4.13}$$

where $v = 2\pi r/P$ was used to determine the velocity of each of the bodies. Setting the centripetal force equal to the gravitational force gives

$$\frac{4\pi^2 m_1 r_1}{P^2} = G\frac{m_1 m_2}{(r_1+r_2)^2}$$
$$\frac{4\pi^2 m_2 r_2}{P^2} = G\frac{m_1 m_2}{(r_1+r_2)^2} \tag{4.14}$$

canceling m_1 and m_2 results in

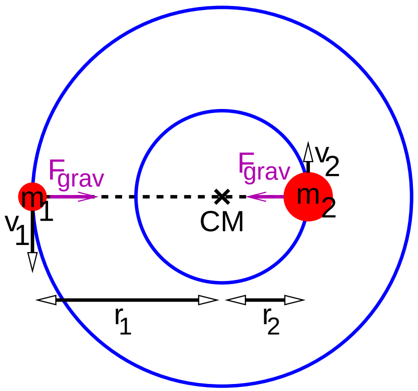
$$\frac{4\pi^2 r_1}{P^2} = G\frac{m_2}{(r_1+r_2)^2}$$
$$\frac{4\pi^2 r_2}{P^2} = G\frac{m_1}{(r_1+r_2)^2} \tag{4.15}$$

Dividing these two equations by each other gives

$$\frac{r_1}{r_2} = \frac{m_2}{m_1} \quad \text{or} \quad m_1 r_1 = m_2 r_2 \tag{4.16}$$

3rd Law

Kepler's 3rd Law: The squares of the periods of the planets, P, are proportional to the cubes of the semimajor axes, a, of their orbits: $P^2 \propto a^3$.



Calculating the motion of two bodies of mass m_1 and m_2 gives Newton's form of Kepler's third law:

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}R^3 \tag{4.11}$$

where $r_1 + r_2 = R$ (for elliptical orbits: R is the semi-major axis).

This is the definition of the *center of mass*.

The total distance between the two bodies is

$$R = r_1 + r_2 = r_1 + \frac{m_1}{m_2}r_1 = r_1\left(1 + \frac{m_1}{m_2}\right) \tag{4.17}$$

Inserting into one of the equations 4.15 gives

$$\frac{4\pi^2}{P^2} \cdot R \cdot \frac{m_2}{m_1+m_2} = \frac{Gm_2}{R^2} \tag{4.18}$$

such that

$$\frac{4\pi^2}{P^2} = \frac{G(m_1+m_2)}{R^3} \quad \text{or} \quad P^2 = \frac{4\pi^2}{G(m_1+m_2)}R^3 \tag{4.19}$$

This is *Newton's form of Kepler's 3rd law*.

3rd Law

Newton's form of Kepler's 3rd law is the most general form of the law.

However, often shortcuts are possible.

Assume one central body dominates, $m_1 = M \gg m_2$:

$$\frac{P^2}{a^3} = \frac{4\pi^2}{GM} = \text{const.} = k \tag{4.22}$$

So, if we know P and a for one body moving around m_1, can calculate k.

For the Solar System, use Earth:

- $P_\oplus = 1$ year (by definition!)
- $a_\oplus = 1$ AU (Astronomical Unit, 1 AU $= 149.6 \times 10^6$ km)

$\Longrightarrow k = 1\,\mathrm{yr}^2\,\mathrm{AU}^{-3}$

Jupiter: $a_{♃} = 5.2$ AU. What is its period?