N-Body Problem, I

In reality, planets also excert forces onto each other.

Total equation of motion for the i-th object:

$$m_i \ddot{\boldsymbol{r}}_i = -\sum_{k=1}^{N} \frac{G m_i m_k}{r_{i,k}^2} \frac{\boldsymbol{r}_i - \boldsymbol{r}_k}{r_{i,k}} \tag{4.24}$$

$\Longrightarrow$ 3N differential equations of 2nd order, requiring 6N integrations for their solution.

Closed solution only possible for 10 of these (6: from motion of center of mass, 3: conservation of angular momentum, 1: conservation of energy).

Analytic solution: “Perturbation theory”:

1. Assume two body motion around Sun for all planets
2. Evaluate force based on this motion.
3. Update positions with this “perturbation”.
4. Iterate (i.e., goto step 2)

N-Body Problem, II

Perturbation theory yields two kinds of perturbations:

periodic perturbations: Terms containing time in $\sin$- and $\cos$-functions.

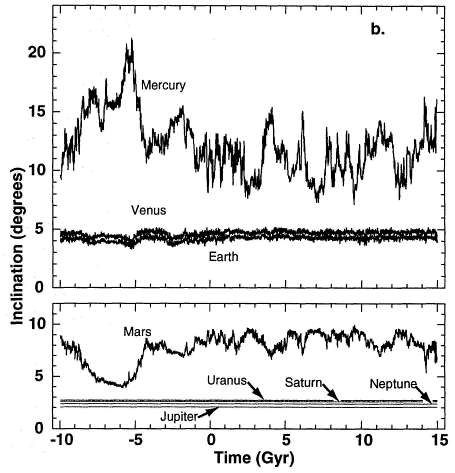
secular perturbations: Long term changes which depend on time (usually as a polynomial).

Analytical approach is very important for understanding the underlying physics, but mathematically *very* tedious. Series do not converge on long timescales (1000’s of years).

$\Longrightarrow$ New high precision calculations are all based on numerical simulations, i.e., direct solution of equation of motion on computers.

Today’s standard: DE102, DE405, DE414 from Jet Propulsion Laboratory, Pasadena, and INPOP06 from Laskar et al., IMCCE, Observatoire de Paris.

Long-Term Evolution of the Solar System



(Laskar, 1994)

Numerical simulations allow to obtain good information about behavior of solar system for timescales of a few 10 Million years around the present $\Longrightarrow$ Important, e.g., for paleoclimatology.

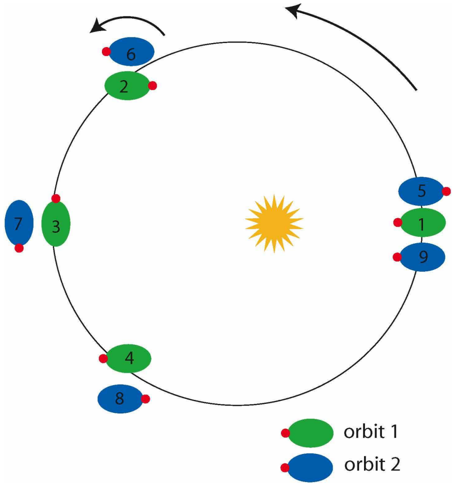
Laskar (1989, 1990): Motion of inner planets is chaotic.

“Chaotic”: Initial errors get amplified exponentially, here by factor of 10 on time scales of ~10 million years.

Important, e.g., for climate variations on Earth (“Milankovitch cycles”).

Also found with different methods by Wisdom and Suskind.

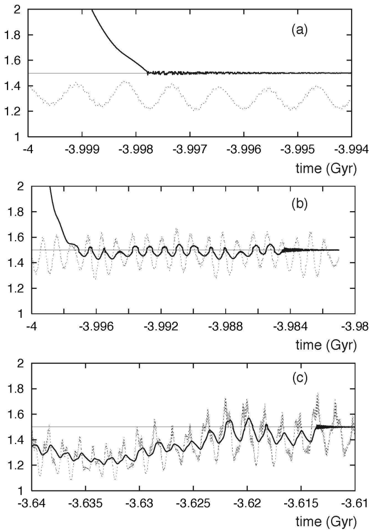
Long-Term Evolution of the Solar System



(courtesy J. Laskar/CNRS)

Rotation period and orbital period of Mercury are in a 3:2 resonance.

Long-Term Evolution of the Solar System



Chaotic motion of Mercury's orbit increases probability of capturing Mercury in its 3:2 resonance with its orbit via tidal dissipation from $<5\%$ in classical theory to $\sim 55\%$.

Similar explanation also for retrograde rotation of Venus, Earth is saved from such a behavior because of the stabilizing influence of the Moon.

(Correia & Laskar, 2004)

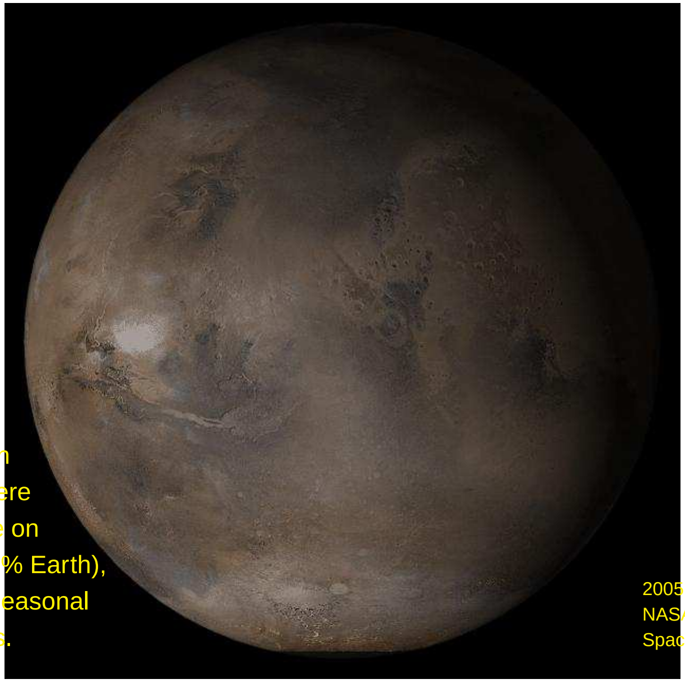
Correia, A. C. M., & Laskar, J., 2004, Nature, 429, 848

Laskar, J., 1989, Nature, 338, 237

Laskar, J., 1990, Icarus, 88, 266

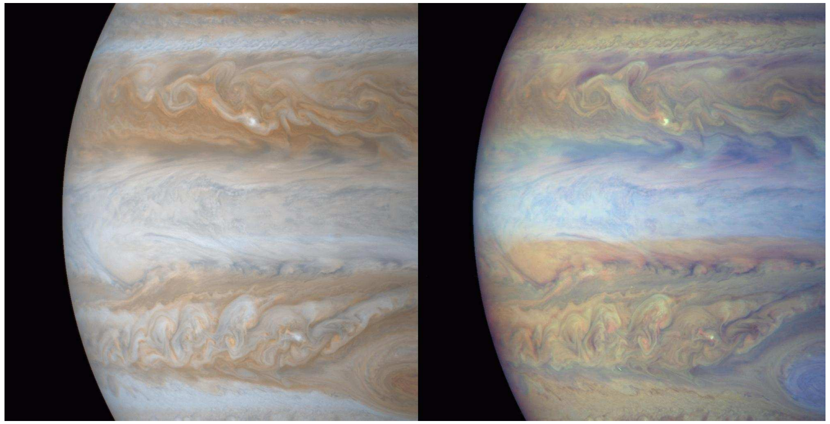
Laskar, J., 1994, A&A, 287, L9

Planets: Atmospheres



Mars: thin atmosphere (pressure on surface 1% Earth), but real seasonal variations.

2005 Feb 7, NASA/Malin Space Systems



NASA/JPL, Cassini, 2000 Dec 31

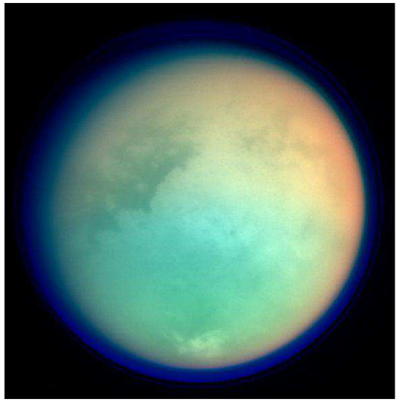
Jupiter: true color image; colors likely from trace content of organic compounds in atmosphere

false color image, red: waterclouds, dark spots: deep hot spots

Overall atmospheric structure: three layers:

Ammonia – ammonia hydrosulfide (NH_4HS) – water ice/water (deepest)

NASA Voyager



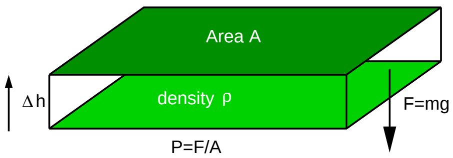
27.10.2004, false colour IR/UV; NASA/ESA

Titan: dense atmosphere, 99% nitrogen, 1% methane, some hydrocarbons, thought to be similar to primordial atmosphere of Earth.

Radius: 2575 km ($\sim$ Mercury!)

ESA probe *Huygens* landed on Titan on 2005 January 14

The structure of an atmosphere is defined by the concept of hydrostatic equilibrium:



The force excerted by gas with density ρ sitting on top of an area A is given by

$$F = mg = \rho V g = A h \rho g \tag{5.1}$$

Such that pressure becomes

$$P = \frac{F}{A} = \rho h g \tag{5.2}$$

where g is the gravitational acceleration.

For a thin atmosphere (g constant): Decrease of P when going upwards by Δh:

$$\Delta P = -\rho g \Delta h \quad \text{and for } \lim_{\Delta h \to 0}: \quad \frac{\mathrm{d}P}{\mathrm{d}h} = -\rho g \tag{5.3}$$

To solve this differential equation, we need a relationship between density and pressure ("equation of state"). For an "ideal gas", this relationship is given by

$$P = (\rho/\mu) k T \tag{5.4}$$

where T is the Temperature (K!), μ the average mass of a gas particle, and k is Boltzmann's constant ($k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$). Therefore:

$$\frac{\mathrm{d}P}{\mathrm{d}h} = -\left(\frac{\mu g}{kT}\right) P \tag{5.5}$$

In order to obtain P as a function of height, h, we need to solve this differential equation with the boundary condition that for $h = 0$, $P = P_0$. This can be done easily using the technique of "separation of variables", assuming that T does not change.

First, divide by P and integrate both sides of the equation with respect to height:

$$\int_0^h \frac{1}{P} \frac{\mathrm{d}P}{\mathrm{d}h} \,\mathrm{d}h = -\int_0^h \left(\frac{\mu g}{kT}\right) \mathrm{d}h$$

We can now substitute $P(h)$ for h on the left hand side. Using the chain rule gives

$$\int_{P_0}^{P(h)} \frac{\mathrm{d}P'}{P'} = -\int_0^h \left(\frac{\mu g}{kT}\right) \mathrm{d}h$$

such that

$$\ln\left(\frac{P(0)}{P(h)}\right) = -\left(\frac{\mu g}{kT}\right) h$$

and exponentiating then gives

$$P(h) = P_0 \exp\left(-\frac{\mu g}{kT} \cdot h\right) = P_0 \exp\left(-\frac{h}{H}\right) \tag{5.6}$$

The pressure in the atmosphere thus decreases exponentially, the characteristic height scale of the decrease is given by the scale height,

$$H = \frac{kT}{\mu g} \tag{5.7}$$

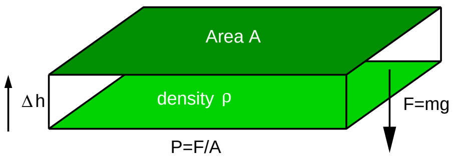
Typical values for the planets are for Earth: $H \sim 9\,\mathrm{km}$.

Remark: The method employed above is called "separation of variables" since people often jump from the first (linear) equation to the third one in one step, by "separating the dependent from the independent variable":

$$\frac{\mathrm{d}P}{\mathrm{d}h} = -\left(\frac{mg}{kT}\right) P \implies \frac{\mathrm{d}P}{P} = -\left(\frac{mg}{kT}\right) \implies \int_{P_0}^{P(h)} \frac{\mathrm{d}P'}{P'} = -\int_0^h \left(\frac{mg}{kT}\right) \mathrm{d}h$$

Hydrostatic Equilibrium

Structure of atmosphere defined through hydrostatic equilibrium:



Force on area A by slab of gas of area A and density ρ:

$$F = mg = \rho V g = Ah\rho g \qquad (5.8)$$

With $P = F/A$ one finds:

$$\frac{\mathrm{d}P}{\mathrm{d}h} = -\rho g$$

where g gravitational acceleration.

Assuming ideal gas, $P = (\rho/\mu)kT$, and isothermal atmosphere:

$$P(h) = P_0 \exp\left(-\frac{\mu g}{kT} \cdot h\right) = P_0 \exp\left(-\frac{h}{H}\right)$$

The pressure in the atmosphere thus decreases exponentially, the characteristic height scale of the decrease is given by the scale height H.

On Earth, $H \sim 9$ km.

Atmospheric Composition

Atmospheric composition of selected terrestrial objects

	Venus	Earth	Mars	Titan
O_2	0.0	21.0	0.0	0.0
H_2O	50ppm	1.0	$<$ 100ppm	0.4ppb
CO_2	96.5	0.0	95.3	10ppb
N_2	3.5	78.0	2.7	90
Ar	0.0	0.9	1.6	

Values are given as percentages by volume.

Titan has atmospheric structure similar to Earth!

after de Pater and Lissauer

Atmospheric Composition

Typical H for the inner planets (Karttunen)

Gas	μ	Venus km	Earth km	Mars km
H_2	2	360	120	290
O_2	32	23	7	81
H_2O	18	40	13	32
CO_2	44	16	5	13
N_2	28	26	8	20
T [K]		275	750	260
g [m s^{-2}]		9.81	8.61	3.77

Atmospheric Composition

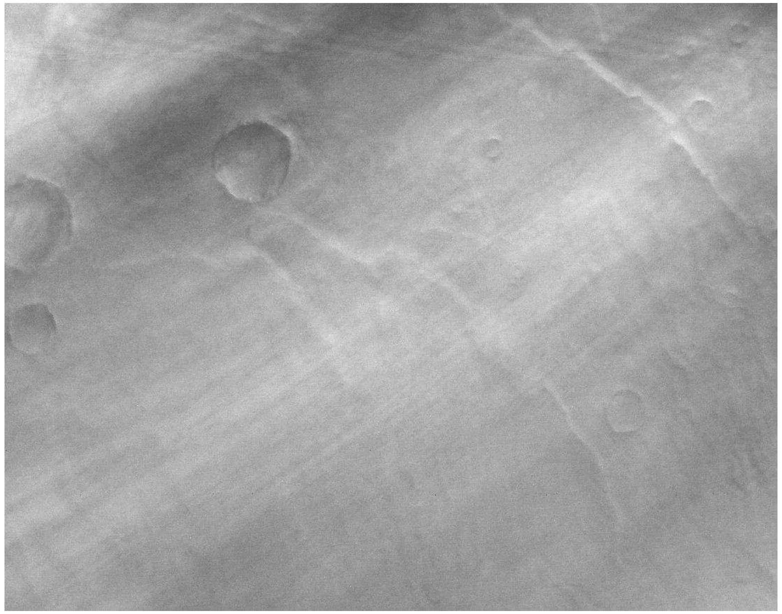
Atmospheric composition of gas giants and the Sun

	Sun	Jupiter	Saturn	Uranus	Neptune
H_2	83.5	86.4	96.3	85±5	85±5
He	19.5	15.7	3.4	18±5	18±5

H: volume percent relative to total atmosphere
He: volume percent relative to H

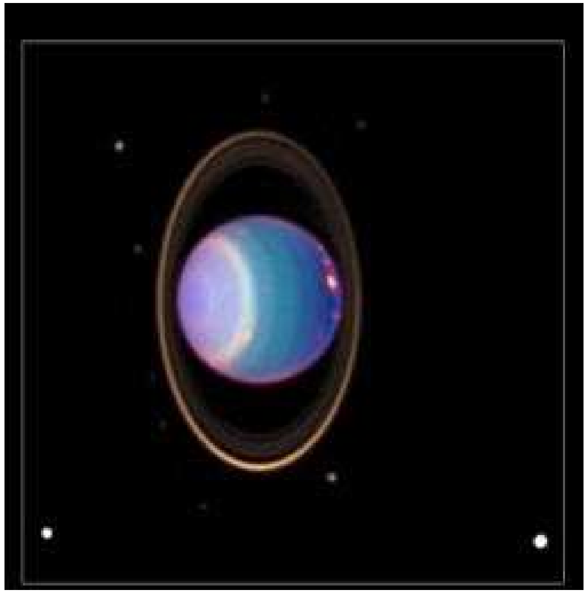
Gas giants have approximately same composition as the Sun.

after de Pater and Lissauer

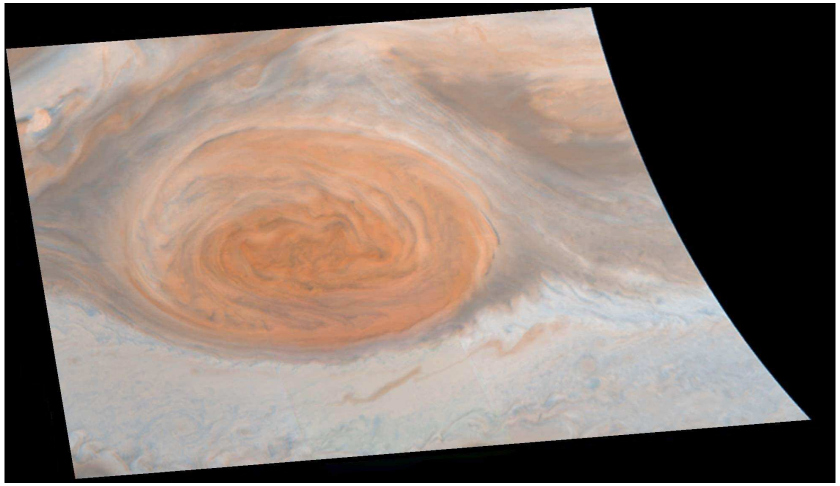


NASA/C.J.Hamilton

Mars: Streaky clouds



HST Image (image enhanced) of Uranus ring system, plus evidence for banded atmosphere and clouds



Great Red Spot
NASA Galileo, 1996 June 26

Jupiter's Great Red Spot: Storm System, $\sim 2\times$ Earth diameter, exists since more than 300 years, 8 km above and 10° cooler than surrounding region (rising high pressure region), rotates counterclockwise (Coriolis force on Southern hemisphere).