The Planets: Dynamics

Introduction, I

Johannes Kepler: Motion of planets governed by three laws:

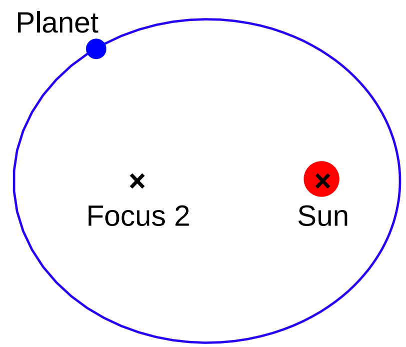
1. Each planet moves in an elliptical orbit, with the Sun at one focus of the ellipse. (“Astronomia Nova”, 1609)
2. A line from the Sun to a given planet sweeps out equal areas in equal times. (“Astronomia Nova”, 1609)
3. The square of the orbital periods of the planets is proportional to the cube of the major axes. (“Harmonice Mundi”, 1619)

Isaac Newton (“Principia”, 1687): Kepler’s laws are consequence of gravitational interaction between planets and the Sun, and the gravitational force is

$$\boldsymbol{F}_1 = -\frac{G m_1 m_2}{r_{12}^2} \frac{\boldsymbol{r}_{21}}{r_{12}} \quad (6.1)$$

where $\boldsymbol{F}_1$ is the gravitational force exerted on object 1, m_1, m_2 are the masses of the interacting objects, r their distance, and $\boldsymbol{r}_{21}/r_{12}$ the unit vector joining the objects, $\boldsymbol{r}_{21} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, $\boldsymbol{r}_{12} = -\boldsymbol{r}_{21}$ and $r_{12} = |\boldsymbol{r}_{12}| = |\boldsymbol{r}_{21}|$.

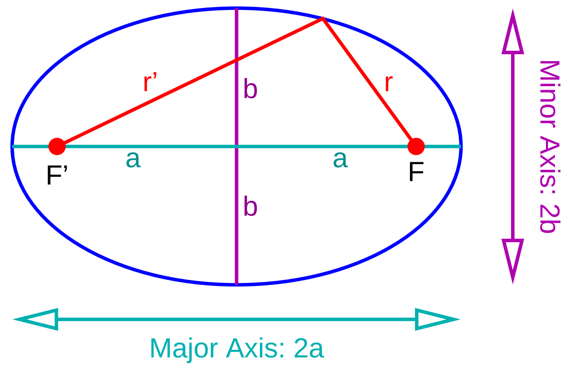
Kepler’s 1st Law, I



Kepler’s 1st Law: The orbits of the planets are ellipses and the Sun is at one focus of the ellipse.

For the planets of the solar system, the ellipses are almost circular, for comets they can be very eccentric.

Kepler’s 1st Law, II

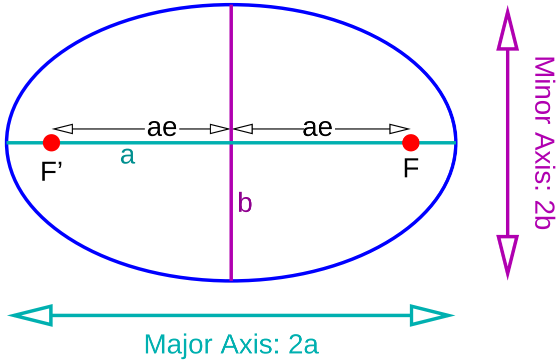


Definition: Ellipse = Sum of distances r, r' from any point on ellipse to two fixed points (foci, singular: focus), F, F', is constant:

$$r + r' = 2a \quad (6.2)$$

where a is called the semi-major axis of the ellipse.

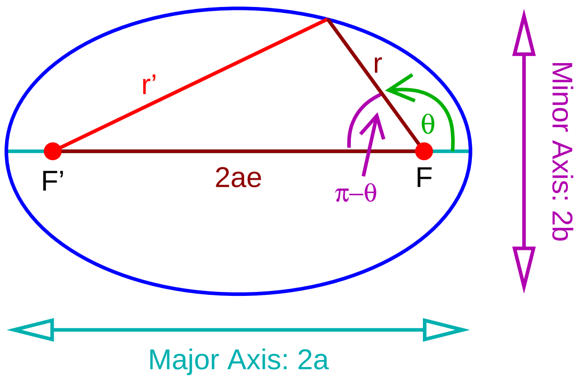
Kepler's 1st Law, III



Definition: Eccentricity e: ratio between distance from centre of ellipse to focal point and semi-major axis.

So circles have $e = 0$.

Kepler's 1st Law, IV



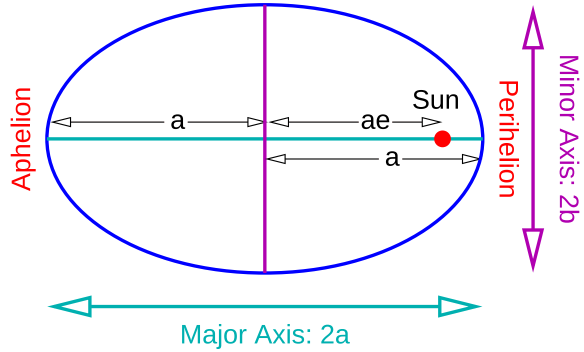
Law of cosines: $r'^2 = r^2 + (2ae)^2 - 2 \cdot r \cdot 2ae \cdot \cos(\pi - \theta)$

use $r + r' = 2a$ and solve for r to find the polar coordinate form of the ellipse:

$$r = \frac{a(1-e^2)}{1 + e\cos\theta} \tag{6.3}$$

Check this for yourself! θ is called the *true anomaly*.

Kepler's 1st Law, V

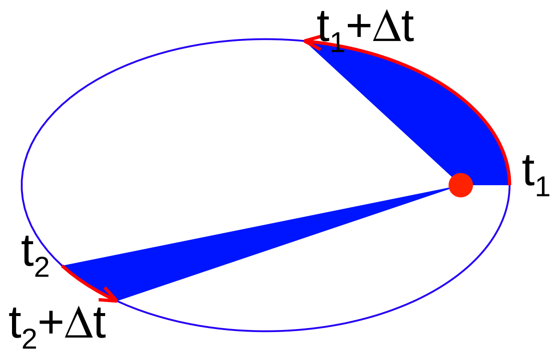


Finally, we need the closest and farthest point from a focus:

$$\begin{aligned} \text{closest point}: d_{\text{perihelion}} &= a - ae = a(1-e) \\ \text{farthest point}: d_{\text{aphelion}} &= a + ae = a(1+e) \end{aligned} \tag{6.4}$$

for stars: periastron and apastron,
for satellites circling the Earth: perigee and apogee.

2nd Law



Kepler's 2nd Law: The radius vector to a planet sweeps out equal areas in equal intervals of time.

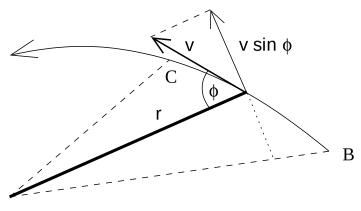
1. Kepler's 2nd Law is also called the *law of areas*.
2. perihelion: planet nearest to Sun $\Longrightarrow$ planet is fastest
3. aphelion: planet farthest from Sun $\Longrightarrow$ planet is slowest

Kepler's 2nd law is a direct consequence of the conservation of angular momentum. Remember that angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v} \tag{6.5}$$

and its absolute value is

$$L = mrv\sin\phi \tag{6.6}$$



To interpret the angular momentum, look at the figure at the left. The projection of the velocity vector perpendicular to the radius vector r is $v\sin\phi$. The distance traveled by the planet in an infinitesimally short time Δt is given by $\Delta x = \Delta t \cdot v\sin\phi$. Therefore, the area of the triangle ABC is given by

$$\Delta A = \frac{1}{2}r\Delta x = \frac{1}{2}r\Delta t v\sin\phi = \frac{L}{2m}\Delta t \tag{6.7}$$

Kepler's 2nd law states that the "sector velocity" $\mathrm{d}A/\mathrm{d}t$ is constant with time:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \frac{L}{2m} = \text{const.} \tag{6.8}$$

To confirm that this claim is true, we need to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2m}\frac{\mathrm{d}L}{\mathrm{d}t} = 0 \tag{6.9}$$

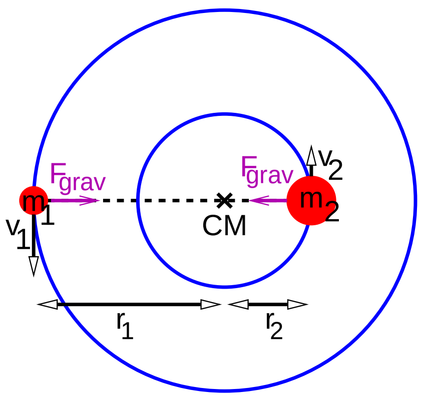
But $\mathrm{d}L/\mathrm{d}t$ is given by

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\times\boldsymbol{p} + \boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v}\times\boldsymbol{p} + \boldsymbol{r}\times\boldsymbol{F} = \boldsymbol{v}\times m\boldsymbol{v} + \boldsymbol{r}\times\frac{GMm}{r^2}\frac{\boldsymbol{r}}{r} = 0 \tag{6.10}$$

since the cross product of a vector with itself is zero. Therefore, Kepler's 2nd law is true and is a consequence of the conservation of angular momentum for a central field.

3rd Law

Kepler's 3rd Law: The squares of the periods of the planets, P, are proportional to the cubes of the semimajor axes, a, of their orbits: $P^2 \propto a^3$.

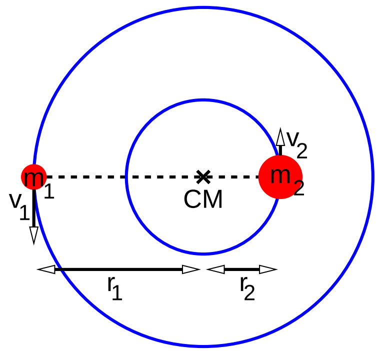


Calculating the motion of two bodies of mass m_1 and m_2 gives Newton's form of Kepler's third law:

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}R^3 \tag{6.11}$$

where $r_1 + r_2 = R$ (for elliptical orbits: R is the semi-major axis).

Circular Motion



For an interpretation of Kepler's third law, consider the motion of two bodies with masses m_1 and m_2 on circular orbits with radii r_1 and r_2 around a point CM (see figure).

The reason for doing the computation with circular orbits is that the following discussion will be easier, however, all results from this section also apply to the general case of elliptical motion. This will be proven later in the lectures on Theoretical Mechanics.

The attractive force between the two points is given by Newton's law:

$$F_{\text{grav}} = G\frac{m_1 m_2}{R^2} = G\frac{m_1 m_2}{(r_1+r_2)^2} \tag{6.12}$$

In order to keep the two bodies on circular orbits, the gravitational force needs to be equal the centripetal force keeping each body on its circular orbit.

The centripetal force is

$$F_{\text{cent},1} = \frac{m_1 v_1^2}{r_1} = \frac{4\pi^2 m_1 r_1}{P^2} \qquad F_{\text{cent},2} = \frac{m_2 v_2^2}{r_2} = \frac{4\pi^2 m_2 r_2}{P^2} \tag{6.13}$$

where $v = 2\pi r/P$ was used to determine the velocity of each of the bodies. Setting the centripetal force equal to the gravitational force gives

$$\frac{4\pi^2 m_1 r_1}{P^2} = G\frac{m_1 m_2}{(r_1+r_2)^2} \qquad \frac{4\pi^2 m_2 r_2}{P^2} = G\frac{m_1 m_2}{(r_1+r_2)^2} \tag{6.14}$$

canceling m_1 and m_2 results in

$$\frac{4\pi^2 r_1}{P^2} = G\frac{m_2}{(r_1+r_2)^2} \qquad \frac{4\pi^2 r_2}{P^2} = G\frac{m_1}{(r_1+r_2)^2} \tag{6.15}$$

Dividing these two equations by each other gives

$$\frac{r_1}{r_2} = \frac{m_2}{m_1} \quad \text{or} \quad m_1 r_1 = m_2 r_2 \tag{6.16}$$

This is the definition of the *center of mass*.

The total distance between the two bodies is

$$R = r_1 + r_2 = r_1 + \frac{m_1}{m_2}r_1 = r_1\left(1 + \frac{m_1}{m_2}\right) \tag{6.17}$$

Inserting into one of the equations 5.15 gives

$$\frac{4\pi^2}{P^2}\cdot R\cdot\frac{m_2}{m_1+m_2} = \frac{Gm_2}{R^2} \tag{6.18}$$

such that

$$\frac{4\pi^2}{P^2} = \frac{G(m_1+m_2)}{R^3} \quad \text{or} \quad P^2 = \frac{4\pi^2}{G(m_1+m_2)}R^3 \tag{6.19}$$

This is *Newton's form of Kepler's 3rd law*.

3rd Law

Newton's form of Kepler's 3rd law is the most general form of the law.

However, often shortcuts are possible.

Assume one central body dominates, $m_1 = M \gg m_2$:

$$\frac{P^2}{a^3} = \frac{4\pi^2}{GM} = \text{const.} = k \tag{6.20}$$

So, if we know P and a for one body moving around m_1, can calculate k.

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So, if we know P and a for one body moving around m_1, can calculate k.

For the Solar System, use Earth:

- $P_\oplus = 1$ year (by definition!)
- $a_\oplus = 1$ AU (Astronomical Unit, 1 AU $= 149.6 \times 10^6$ km)

$\Longrightarrow k = 1\,\text{yr}^2\,\text{AU}^{-3}$

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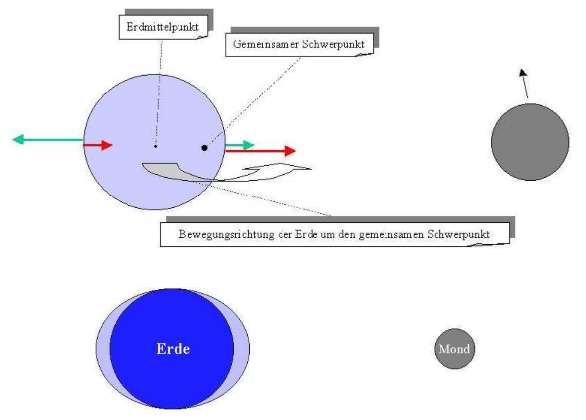
Jupiter: $a_{♃} = 5.2$ AU. What is its period?

Answer: $P_{♃}^2 = 1\,\text{yr}^2\,\text{AU}^{-3} \cdot 5.2^3\,\text{AU}^3 \sim 140\,\text{yr}^2$, or $P_{♃} \sim 12$ years

(with pocket calculator: $P_{♃} = 11.86$ years)

Tidal forces: The Earth–Moon system, I



(Einsteins-Erben)

gravitational force of the moon onto earth

centrifugal force of earth around centre of gravity

Tidal forces: The Earth–Moon system, II

Gravitational acceleration for center of Earth (0) and for the point closest to a gravitating body (1):

$$a_0 = \frac{GM}{r^2} \quad \text{and} \quad a_1 = \frac{GM}{(r - R_\oplus)^2} \tag{6.24}$$

such that difference in acceleration:

$$\Delta a = a_1 - a_0 = \frac{GM}{(r - R_\oplus)^2} - \frac{GM}{r^2} = \frac{GM}{r^2}\left(\frac{1}{(1 - \frac{R_\oplus}{r})^2} - 1\right) \sim \frac{GM}{r^2}\left(1 + 2\frac{R_\oplus}{r} - 1\right) = \frac{2GMR_\oplus}{r^3} \tag{6.25}$$

Therefore the tides due to the Moon are

$$\Delta a_☾ = \frac{2GM_☾ R_\oplus}{r_☾^3} \tag{6.26}$$

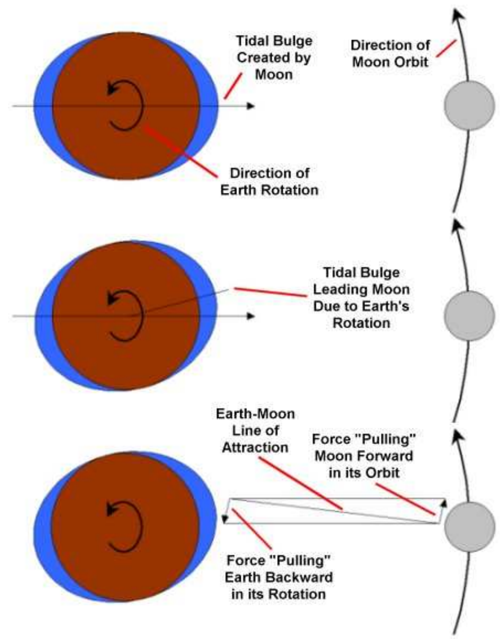
and the tides due to the sun are

$$\Delta a_\odot = \frac{2GM_\odot R_\oplus}{(1.\mathrm{AU})^3} \tag{6.27}$$

But since the mass of the moon is $M_☾ \sim M_\oplus/81$ and the $R_☾ \sim R_\oplus/60$

$\Longrightarrow \Delta a_\odot/\Delta a_☾ = 0.46$

$\Longrightarrow$ moon twice as important as sun; two sets of tidal bulges, spring-tides



Bound Rotation, I

- Tidal bulges preceed rotation by 2.5 h due to friction (ocean ground)
- Rotation of earth slows down $\Longrightarrow$ length of day increases by 1.6 ms century^{-1}
  confirmed by historic solar eclipse observations (e.g., Thales, 585 B.C., eclipse observations by Chinese astronomers)
- 370 Mio. years ago (Devon): 1 Year$\sim$400 days (from coral growth)
- Moon is accelerated, its distance increases by 4 cm yr^{-1}
- length of the month grows
- friction ceases when 1 day = 1 month (synodic) = 50 days (now)
- equilibrium, i.e., bound rotation, is reached in $\sim$1 Billion years
- Moon's rotation bound to orbital motion by tidal friction on the liquid interior of the moon soon after formation
- Other moons are also synchronized: e.g. Galilean moons of Jupiter
- Pluto/Charon both synchronized

Stability of Satellites, I

Tidal forces also important for the stability of small bodies (e.g., moons) moving around a central body at a distance r.

Notation: mass $M_{\mathrm{p,s}}$, radius $R_{\mathrm{p,s}}$, density $\rho_{\mathrm{p,s}}$ where p: planet, s: satellite.

Satellite is bound by its own gravitational force. Satellite is not ripped apart, if binding force $F_{\mathrm{G}} >$ tidal force F_{T}:

Binding force is $\sim$ mutual attraction of two halfs of the satellite separated by R_{s}:

$$F_{\mathrm{G}} = \frac{GM_{\mathrm{s}}M_{\mathrm{s}}}{4R_{\mathrm{s}}^2} > \frac{GM_{\mathrm{p}}M_{\mathrm{s}}R_{\mathrm{s}}}{r^3} = F_{\mathrm{T}} \tag{6.28}$$

where binding force $\sim$ gravitational attractions of two hemispheres of the satellite onto each other.

$$\Longrightarrow \frac{GM_{\mathrm{s}}M_{\mathrm{s}}}{4R_{\mathrm{s}}^2} > \frac{GM_{\mathrm{p}}M_{\mathrm{s}}R_{\mathrm{s}}}{r^3} \Longrightarrow \frac{M_{\mathrm{s}}}{4R_{\mathrm{s}}^3} > \frac{M_{\mathrm{p}}}{r^3} \text{ or } \frac{\rho_{\mathrm{s}}}{4} > \frac{R^3\rho_{\mathrm{p}}}{r^3} \tag{6.29}$$

This means there is a critical distance (Roche, 1850):

$$\frac{r_{\mathrm{c}}}{R} = \sqrt[3]{\frac{4\rho_{\mathrm{p}}}{\rho_{\mathrm{s}}}} \quad \frac{r_{\mathrm{c}}}{R} = 2.44\sqrt[3]{\frac{\rho_{\mathrm{p}}}{\rho_{\mathrm{s}}}} \tag{6.30}$$

$\Longrightarrow$ If of same density, the moon has to be more distant than 2.44 planetary radii from the planet in order to avoid disruption. This is important, e.g., for the formation of rings.

Precession and Nutation

Earth is $\sim$ rotational ellipsoid, orbits of Sun and Moon are *not* in plane of equator

(Earth's axis has tilt of $\sim$23.5°, moon's orbit tilted by 7° against ecliptic)

$\Longrightarrow$ Sun and Moon excert torques onto Earth

Earth's rotational axis is not stable in space.

Two major effects:

luni-solar precession: Earth's axis rotates around pole of ecliptic once every 25800 years ($\sim$50″ per year).

Already discovered by Hipparcos in $\sim$200 BC!

nutation: "Wobble" with $\sim$18 year periodicity caused by short-term perturbations caused by Moon and Sun.

N-Body Problem

Apart from Sun, motion of planets also influenced by forces between planets:

Total equation of motion for the i-th object:

$$m_i\ddot{\boldsymbol{r}}_i = -\sum_{k=1}^{N}\frac{Gm_im_k}{r_{i,k}^2}\frac{\boldsymbol{r}_i - \boldsymbol{r}_k}{r_{i,k}} \tag{6.31}$$

$\Longrightarrow$ 3N differential equations of 2nd order, requiring 6N integrations for their solution.

Closed solution only possible for 10 of these (6: from motion of center of mass, 3: conservation of angular momentum, 1: conservation of energy).

Analytic solution: "Perturbation theory":

1. Assume two body motion around Sun for all planets
2. Evaluate force based on this motion.
3. Update positions with this "perturbation".
4. Iterate (i.e., goto step 2)

N-Body Problem

Perturbation theory yields two kinds of perturbations:

periodic perturbations: Terms containing time in $\sin$- and $\cos$-functions.

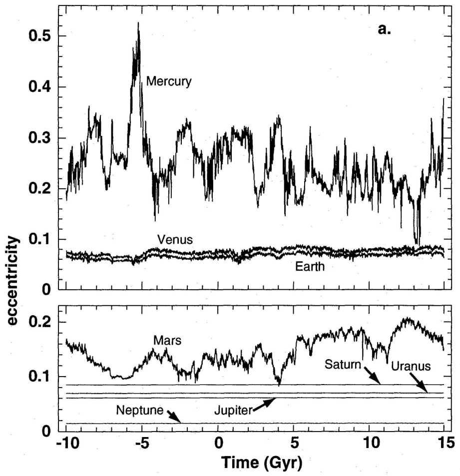
secular perturbations: Long term changes which depend on time (usually as a polynomial).

Analytical approach is very important for understanding the underlying physics, but mathematically *very* tedious. Series do not converge on long timescales (1000's of years).

$\Longrightarrow$ New high precision calculations are all based on numerical simulations, i.e., direct solution of equation of motion on computers.

Today's standard: DE102, DE405, DE414 from Jet Propulsion Laboratory, Pasadena, and INPOP06 from Laskar et al., IMCCE, Observatoire de Paris.

Long-Term Evolution of the Solar System



(Laskar, 1994)

Numerical simulations allow to obtain good information about behavior of solar system for timescales of a few 10 Million years around the present $\Longrightarrow$ Important, e.g., for paleoclimatology.

Laskar (1989, 1990): Motion of inner planets is chaotic.

"Chaotic": Initial errors get amplified exponentially, here by factor of 10 on time scales of $\sim$10 million years.

Important, e.g., for climate variations on Earth ("Milankovitch cycles").

Also found with different methods by Wisdom and Suskind.

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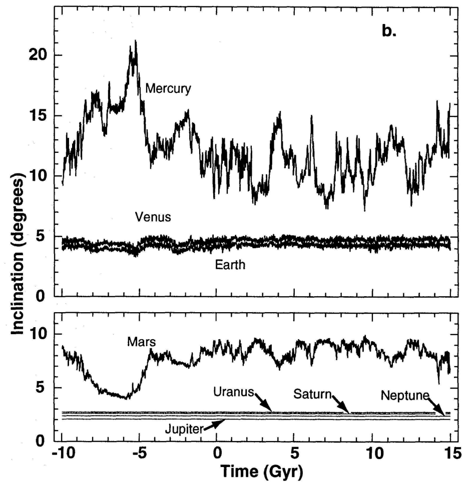

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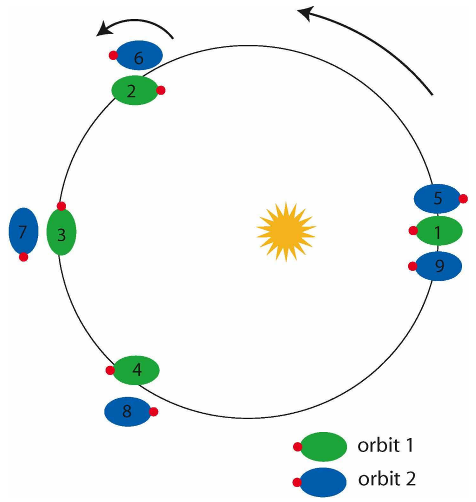
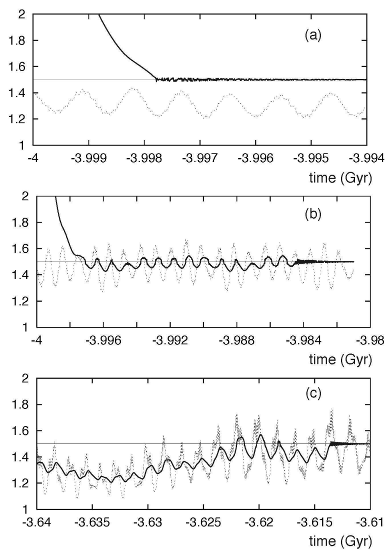


(courtesy J. Laskar/CNRS)

Rotation period and orbital period of Mercury are in a 3:2 resonance.

Long-Term Evolution of the Solar System



Chaotic motion of Mercury's orbit increases probability of capturing Mercury in its 3:2 resonance with its orbit via tidal dissipation from $<$5% in classical theory to $\sim$55%.

Similar explanation also for retrograde rotation of Venus, Earth is saved from such a behavior because of the stabilizing influence of the Moon.

(Correia & Laskar, 2004)

Correia, A. C. M., & Laskar, J., 2004, Nature, 429, 848

Laskar, J., 1989, Nature, 338, 237

Laskar, J., 1990, Icarus, 88, 266

Laskar, J., 1994, A&A, 287, L9