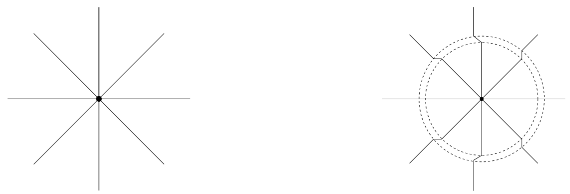
One of the most important results of classical electrodynamics is the fact that accelerated charges emit radiation. Here, we will derive the formula describing this result, "Larmor's formula", using a classical treatment due to J.J. Thomson and revived by Malcolm Longair. The *ab initio* derivation from Maxwell's equations gives the same result, but after a significantly more complicated calculation. Please see textbooks on electrodynamics for this.

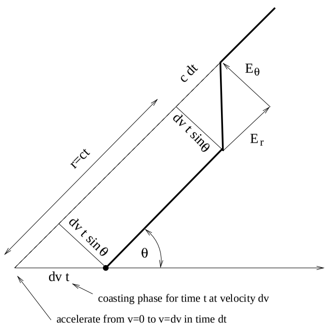
We start by taking a stationary charge at rest at time $t = 0$. The field lines from that charge have the simple configuration shown on the left side of the figure below.

We now accelerate the charge by a velocity difference Δv for a time interval Δt and look at the field line configuration at a time t after this.



Outside of a sphere of radius $r \sim ct$ the field lines still point towards the original position of the charge, because the information about the acceleration has not yet moved farther out than r. Inside of this sphere the field lines point towards the location the charge had after the acceleration. Properly spoken, the figure is not correct as the charge continued to drift with its new velocity since the end of the acceleration. The field lines in the figure should reflect this, but given that that information has not yet propagated out to r either, we can ignore it, and all of the computations we do below will only need information local ti r. Since the electric field lines have to be connected, there is a small region of width $c\Delta t$ in which the electric field has a non-radial component. An observer in this region will measure a temporal change in the E-field strength. By definition, a time dependent E-field is electromagnetic radiation and therefore this *Gedankenexperiment* has shown us that accelerating a charge will always lead to electromagnetic radiation!

To obtain the power radiated by the accelerated charge we need to know the strength of the electric field in the region of the electromagnetic pulse. This can be done by considering the following sketch of one E-field line:



where, for simplicity $\mathrm{d}t = \Delta t$ and $\mathrm{d}v = \Delta v$.

From simple geometry, one finds for the ratio of the electric fields in the pulse region:

$$\frac{E_\theta}{E_r} = \frac{\Delta v t \sin\theta}{c\Delta t} \tag{4.54}$$

E_r follows from Coulomb's law:

$$E_r = q \cdot \frac{1}{r^2} \tag{4.55}$$

and because we're observing at time t:

$$r = ct \tag{4.56}$$

Inserting E_r and r into Eq. (4.54) gives

$$E_\theta = E_r \frac{\Delta v}{\Delta t}\frac{t \sin\theta}{c} = q \frac{1}{r}\frac{1}{ct}\frac{\Delta v}{\Delta t}\frac{t\sin\theta}{c} \tag{4.57}$$

In the limit $\Delta t \to 0$, we can identify $\Delta v/\Delta t$ with the acceleration $\dot{v}$ (where the dot denotes differentiation with respect to time), and therefore we finally obtain

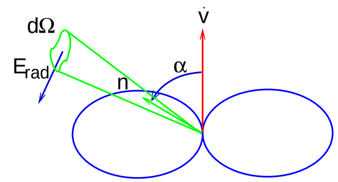
$$E_\theta = \frac{q}{rc^2}\dot{v}\sin\theta \tag{4.58}$$

So, during the pulse the E-field in the θ-direction changes from 0 to E_θ and back to 0. This is a pulse of electromagnetic radiation.

The energy flow per unit area and per second, S, is then given from Poynting's theorem as

$$S = \frac{c}{4\pi}E^2 = \frac{q^2}{4\pi c^3 r^2}\dot{v}^2 \sin^2\theta \tag{4.59}$$

This means that the energy loss has *dipolar form*, as shown in the following figure. Often, this equation is called *Larmor's formula* (although we will encounter another Larmor's formula below).



Note that the energy loss is symmetric, this means that the radiating particle is only loosing energy, but not momentum.

To obtain the total energy lost by the particle, we need to integrate the energy lost over all directions

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{4\pi\mathrm{sr}} S\, r^2\, \mathrm{d}\Omega = \int_{4\pi\mathrm{sr}} \frac{q^2}{4\pi c^3 r^2}\dot{v}^2 \sin^2\theta\, r^2\, \mathrm{d}\Omega \tag{4.60}$$

where $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the surface element of a sphere, θ and φ are the usual spherical coordinates, with θ going from 0 to π and φ from 0 to 2π. The total surface area of a sphere is 4π steradians. Performing the integral, lumping all constants into a helper variable A gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} = A\int_{4\pi\mathrm{sr}} \sin^2\theta\; \mathrm{d}\Omega = A\int_0^\pi \sin^2\theta\, 2\pi\, \sin\theta\; \mathrm{d}\theta = 2\pi A\int_0^\pi \sin^3\theta\; \mathrm{d}\theta = \frac{8\pi}{3}A \tag{4.61}$$

And therefore we obtain *Larmor's formula* for the energy loss of an accelerated charge:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{8\pi}{3}\frac{q^2}{4\pi c^3}\dot{v}^2 = \frac{2q^2}{3c^3}\dot{v}^2 \tag{4.62}$$

Measurements and Statistics

Introduction

Scientific Measurements: precision with which quantity is measured is important, strongly depends on uncertainty of measurements

Examples:

- Characterization of a measurement: mean, standard deviation,. . .
- Variation of sources: two measurements give different numbers for the brightness of a source, but is it really different)?
- Hypothesis testing: does a set of measurements (dis)agree with a hypothesis

Will give a practical overview on most common issues here.

Literature:

- Bradt, 2004, Astronomy Methods, Cambridge: CUP
- Bevington, Robinson, 1992, Data Reduction and Error Analysis for the Physical Sciences, New York: McGraw-Hill
- Roe, 2001, Probability and Statistics in Experimental Physics, New York, Heidelberg: Springer

Only a small selection of the most application oriented books is given here, for details there is a whole body of useful literature around. . .

Binomial Coefficient

There are n distinguishable objects are in a box. How many ways, n_r, are there to take r objects out of the box?

1. First trial: n different possibilities.
2. Second trial: $n - 1$ different possibilities.
3. rth trial: $n - r + 1$ different possibilities.

Therefore

$$n_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} \tag{5.1}$$

where

$$n! = n(n-1)(n-2)\cdots 1, \quad 0! := 1 \tag{5.2}$$

Binomial Coefficient

Modify previous question: How many *different* sets of r objects can be picked from a box containing n distinguishable objects? (this is number is written $\binom{n}{r}$)

Example: Assume we have 2 Objects.

Can pick object 1, then object 2, *or* object 2, then object 1. $\Longrightarrow \binom{2}{2} = 2$.

Answer:

$$\binom{n}{r} = \frac{n_r}{\text{number of ways of picking the same } r \text{ objects}} \tag{5.4}$$

Since there are $r!$ different ways to pick the same r objects, this means that the binomial coefficient is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \tag{5.5}$$

Binomial Distribution

Bernoulli trials: Repeated *independent* trials with two possible outcomes.

Examples: coin tossing, decay of particle into one of two different decay products,. . .

Let:

- p: probability of success
- $q = 1 - p$: probability of failure

Question: What is probability, $P_{n,r}$, of r successes in n trials?

Answer: Since probabilities multiply, the probability to obtain a certain combination of r successes *once* is $p^r q^{n-r}$. Since there are $\binom{n}{r}$ ways of obtaining this combination,

$$P_{n,r} = \binom{n}{r} p^r q^{n-1} = \frac{n!}{r!(n-r)!} p^r q^{n-r} \tag{5.7}$$

Example (after Reif): $n = 80$ molecules in a box, $p = 0.5$ a molecule is on the left or the right half of the box. The probability that only $r = 1$ molecule is in one half of the box is $P_{80,1} = 6 \times 10^{-23}$, i.e., if we check every $\Delta t = 10^{-6}$ s, we will see this happening less than once in 15×10^{9} years.

Poisson Distribution

One important limit of the binomial distribution is the Poisson distribution, valid if

$$p \ll 1 \quad \text{and} \quad n \gg 1 \quad \text{and} \quad r \ll n$$

Writing the number of events $\lambda = pn$, the binomial distribution can be written

$$P_{n,r} = \frac{n!}{r!(n-r)!} p^r q^{n-r} \tag{5.11}$$

$$= \frac{n!}{r!(n-r)!} \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r} \tag{5.12}$$

$$= \frac{\lambda^r}{r!} \underbrace{\frac{n!}{(n-r)!n^r}}_{T_1} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-r}}_{T_2} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{n}}_{T_3} \tag{5.13}$$

In the limit $n \to \infty$ the terms $T_1, \dots, T_3$ are:

$$T_1 = \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} \cdots \frac{n-r+1}{n} \to 1 \tag{5.14}$$

$$T_2 = \left(1 - \frac{\lambda}{n}\right)^{-r} \to 1 \tag{5.15}$$

$$T_3 = \left(1 - \frac{\lambda}{n}\right)^{n} \to r^{-\lambda} \quad \text{by definition of } e^{-\lambda}! \tag{5.16}$$

Poisson Distribution

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Writing the number of events $\lambda = pn$, the binomial distribution can be written

$$P_{n,r} = \frac{n!}{r!(n-r)!} p^r q^{n-r} \tag{5.17}$$

$$= \frac{n!}{r!(n-r)!} \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r} \tag{5.18}$$

$$= \frac{\lambda^r}{r!} \underbrace{\frac{n!}{(n-r)!n^r}}_{T_1} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-r}}_{T_2} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{n}}_{T_3} \tag{5.19}$$

Therefore, for $n \to \infty$ and $p \ll 1$, the limit of the binomial distribution is the Poisson distribution,

$$P_r = \frac{\lambda^r}{r!} e^{-\lambda} \tag{5.20}$$

Interpretation: P_r = Probability of detecting r events in a measurement when the average over a large number of such measurements is λ.

Alternative Derivation

The most important application of the Poisson distribution is in photon counting applications:

Let the probability to detect *one* event in a time interval Δt be

$$P_1(\Delta t) = \mu \Delta t \tag{5.21}$$

Let $P_r(t)$ the probability that exactly r events have been detected in time interval t. Then

$$P_r(t + \Delta t) = P_r(t)P_0(\Delta t) + P_{r-1}(t)P_1(\Delta t) \tag{5.22}$$

$$= P_r(t)(1 - \mu\Delta t) + P_{r-1}(t)\mu\Delta t \tag{5.23}$$

$$= P_r(t) - P_r(t)\mu\Delta t + P_{r-1}(t)\mu\Delta t \tag{5.24}$$

such that

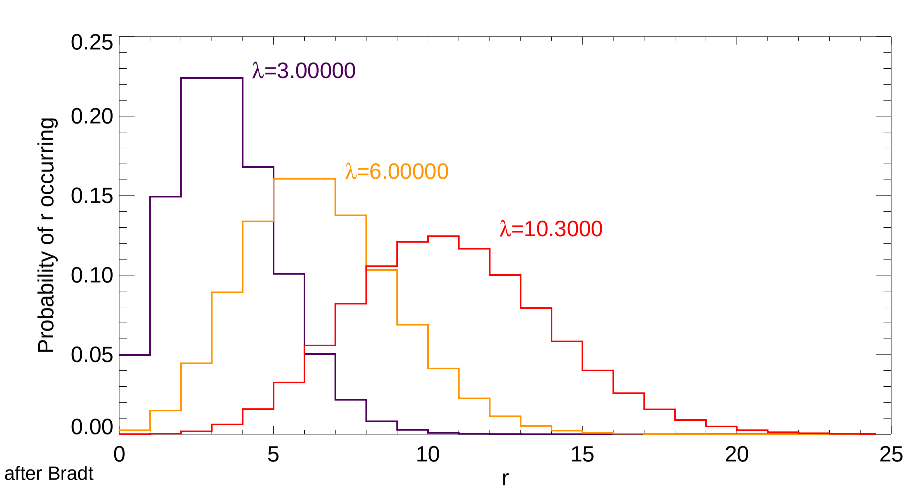
$$\frac{P_r(t+\Delta t) - P_r(t)}{\Delta t} = -\mu P_r(t) + \mu P_{r-1}(t) \quad \Longrightarrow \quad \frac{dP_r}{dt} = -\mu P_r + \mu P_{r-1} \tag{5.25}$$

where $P_{-1} = 0$. For $r = 0$ the solution of the differential equation is $P_0 = e^{-\mu t}$, for higher r, a proof by induction gives the Poisson distribution:

$$P_r = \frac{(\mu t)^r}{r!} e^{-\mu t} \tag{5.26}$$

(set $\lambda = \mu t$!).

Alternative Derivation



Poisson distribution for different λ. Note asymmetry for low λ!

Normal Distribution

Statistical distributions such as Poisson are generally strongly peaked around some maximum $\tilde{r}$

$\Longrightarrow$ describe P_r approximately by a smooth function

Since P_r strongly peaked, it makes sense to expand its *logarithm*:

$$\log P_r = \log P_{\tilde{r}} + \left.\frac{d\log P}{dr}\right|_{\tilde{r}} \frac{r-\tilde{r}}{1!} + \left.\frac{d^2\log P}{dr^2}\right|_{\tilde{r}} \frac{(r-\tilde{r})^2}{2!} + \cdots \quad (5.30)$$

Since P has a maximum at $\tilde{r}$, the first derivative is zero and the second derivative is negative. Therefore

$$P_r \sim P_{\tilde{r}} e^{-|d^2\log P/dr^2|(r-\tilde{r})^2/2} \quad (5.31)$$

Now define the probability density function, f, such that

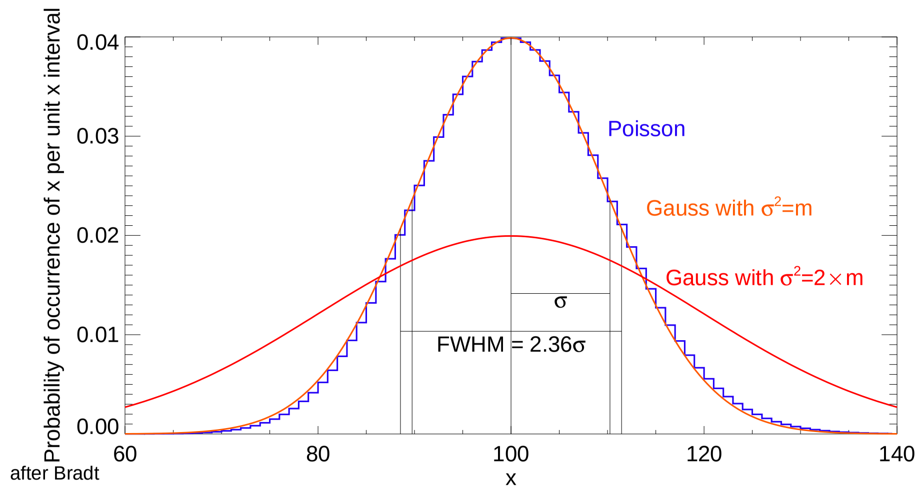
$$P_n \Delta n \sim f dx \quad (5.32)$$

where $x = n\ell$ is defined in terms of a step length ℓ (i.e., $dx = \ell\Delta n$) to go from discrete to continuous variables, to find the Gaussian normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\tilde{x})^2/2\sigma^2} \quad (5.33)$$

where $\sigma^2 = \ell^2/|d^2\log P/dr^2|$ and $\tilde{x} = \tilde{n}\ell$.

Normal Distribution



Gauss with $\sigma^2 = \tilde{x}$ and $\sigma^2 = 2\tilde{x}$, and Poisson (assymetry!).

Normal Distribution

To characterize distributions, we use different quantities:

- most probable value: maximum of f
- probability of finding random variable X between x_1 and x_2:

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)dx \quad (5.34)$$

Gaussian: $\tilde{x} \pm \sigma = 68\%$, $\tilde{x} \pm 2\sigma = 4.6\%$, $\tilde{x} \pm 3\sigma = 0.27\%$

- expectation value:

$$E(X) = \int_{-\infty}^{+\infty} x f(x)dx \quad (5.35)$$

Capital letters designate the random variable...

- The nth moment is defined as

$$E(X^n) = \int_{-\infty}^{+\infty} x^n f(x)dx \quad (5.36)$$

Normal Distribution

Two moments are especially important:

1. The mean is the expectation value of x:

$$\langle x\rangle = E(x) = \int_{-\infty}^{+\infty} x f(x)dx \tag{5.37}$$

Gaussian: $\langle x\rangle = \tilde{x}$, Poisson: $\langle x\rangle = \lambda$

2. The variance is

$$\mathrm{Var}(X) = E\left((X-E(X))^2\right) = \int_{-\infty}^{+\infty} (x-E(X))^2 f(x)dx \tag{5.38}$$

Gaussian: $\mathrm{Var}(X) = \sigma^2$, Poisson: $\mathrm{Var}(X) = \lambda$

3. The standard deviation is the square root of the variance:

$$\mathrm{StdDev}(X) = \sqrt{\mathrm{Var}(X)} \tag{5.39}$$

Gaussian: $\mathrm{Var}(X) = \sigma$, Poisson: $\mathrm{Var}(X) = \lambda^{1/2}$

Example

Let's look at photon counting detector such as a CCD and larger event numbers

$\Longrightarrow$ can approximate Poisson distribution by Gaussian

Assume average number of detected photons per pixel is $m = 100$

$\Longrightarrow \sigma = m^{1/2} = 10$

$\Longrightarrow$ probability of being outside the 70–130 photon range is 0.27%, or 1:370.

Let's assume a source is brighter than 130 photons, then the chance that this brightening is random is 1:370.

Is this enough? Probably not

$\Longrightarrow$ increase to 4σ or 5σ ($= 6 \times 10^{-7}$).

Bradt: It is sometimes tempting to report such a result as real if a repeat measurement is difficult or impossible… The temptation is even greater if the indicated result is of sufficient importance to win one great fame. Because of this one must always try to repeat the measurement or at least take great care to understand all the factors that went into the probability calculation that makes such an event seem real. One can also earn fame as a fool by over interpreting data.

Error Propagation

Inferred quantities are often derived from measured quantities by some formula:

$$\xi = f(x_1, x_2, \ldots, x_n) \tag{5.40}$$

where x_1, x_2,…,x_n are measured quantities.

The uncertainty of ξ, σ_ξ, is then given by Gaussian error propagation:

$$\sigma_\xi^2 = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 \tag{5.41}$$

where $\sigma_{x_i}^2$ is the variance of x_i.

Remark: This assumes that x_1, x_2,… are *independent variables*, which is often *not* the case!

Derivation: Taylor.

Example: If $\xi = x + y$ then $\sigma_\xi^2 = \sigma_x^2 + \sigma_y^2$ ("errors are added in quadrature").

Background Subtraction

In astronomy, sources are often faint $\Longrightarrow$ instrumental background becomes very important.

Assume measurement has duration Δt, during this time S counts arrive from source, B counts from background, i.e., on-source measurement yields $S + B$ counts, and off-source measurement yields B counts.

To obtain source flux, need to calculate

$$S = (S + B) - B \tag{5.42}$$

Variance on this difference is

$$\sigma_S^2 = \sigma_{S+B}^2 + \sigma_B^2 = S + B + B = S + 2B \tag{5.43}$$

since $\sigma_{S+B} = \sqrt{S+B}$, $\sigma_B = \sqrt{B}$ (Poisson!).

Therefore the significance of a source detection is

$$\frac{S}{\sigma_S} = \frac{S}{\sqrt{S+2B}} \tag{5.44}$$

and the measured count rate and its uncertainty is

$$r_S = \frac{S}{\Delta t} \pm \frac{\sigma_S}{\Delta t} \tag{5.45}$$

Low Background

Assume that background is low, then

$$\frac{S}{\sigma_S} \sim \frac{S}{\sqrt{S}} = \sqrt{S} = \sqrt{r_S \Delta t} \tag{5.46}$$

The significance of an observation increases only with the square-root of the observing time.

A source is seen at 2σ in time Δt. What is the exposure ΔT needed to see source at 5σ?

High Background

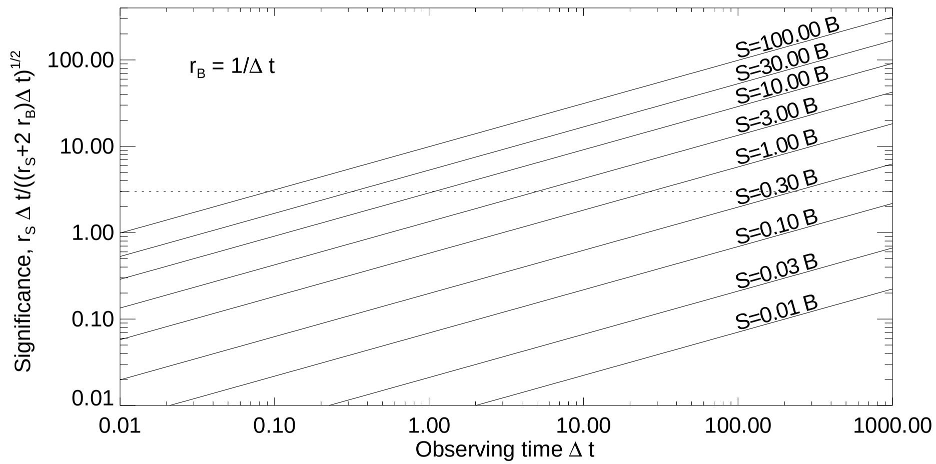
If background is high, $B \gg S$, then

$$\frac{S}{\sigma_S} \sim \frac{S}{\sqrt{2B}} = \frac{r_S \Delta t}{\sqrt{2 r_B \Delta t}} = \frac{r_S}{\sqrt{2 r_B}} \sqrt{\Delta t} \tag{5.49}$$

In the high background case the significance of a detection also increases with the square root of the observing time.

. . . but the sensitivity is much less than in the case of the low background detector.

Summary



Time Δt needed to detect source at a certain significance, for a background countrate of $r_B = 1/\Delta t$, and different source strengths.