



### Question 1: *Properties of the Planck Spectrum*

a) Planck's formula in *frequency space* was shown in the lectures to be

$$\frac{dE}{dAdtd\Omega d\nu} = B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \quad (3.18)$$

Note that

$$\frac{dE}{dAdtd\Omega} = B_\nu d\nu \quad (w1.1)$$

which is independent of frequency. It is easy to see that a similar relationship has to hold for  $B_\lambda$ , such that

$$|B_\nu d\nu| = |B_\lambda d\lambda| \quad (w1.2)$$

Using the previous equation, show that in *wavelength space*

$$B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT) - 1} \quad (w1.3)$$

Since  $\lambda\nu = c$

$$\frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2} \iff d\nu = -\frac{c}{\lambda^2} d\lambda \quad (s1.1)$$

For this reason

$$\begin{aligned} |B_\nu d\nu| &= \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \frac{c}{\lambda^2} d\lambda \\ &= \frac{2hc^3}{\lambda^3 c^2} \frac{1}{\exp(hc/\lambda kT) - 1} \frac{c}{\lambda^2} d\lambda \\ &= \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1} d\lambda \quad (s1.2) \end{aligned}$$

b) *Derivation of the Rayleigh-Jeans-Law*: Using

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + \dots \quad (3.48)$$

show that

$$B_\nu \approx \frac{2\nu^2}{c^2} kT \quad (3.49)$$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \sim \frac{2h\nu^3}{c^2} \frac{1}{1 + \frac{h\nu}{kT} - 1} = \frac{2h\nu^3}{c^2} \frac{kT}{h\nu} = \frac{2\nu^2}{c^2} kT \quad (s1.3)$$

c) *Derivation of the Wien spectrum:* Using

$$\exp\left(\frac{h\nu}{kT}\right) - 1 \sim \exp\left(\frac{h\nu}{kT}\right) \quad (3.51)$$

show that for  $h\nu \gg kT$

$$B_\nu \sim \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{kT}\right) \quad (3.52)$$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \sim \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT)} = \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{kT}\right) \quad (s1.4)$$

d) The energy density for radiation was found to be

$$u_\nu(\Omega) = \frac{1}{c} I_\nu \quad (2.71)$$

i. Show that for isotropic radiation

$$u_\nu = \int_{4\pi \text{ sr}} u \, d\Omega = \frac{4\pi}{c} I_\nu \quad (w1.5)$$

Since the radiation is isotropic, we have

$$u_\nu = \int_{4\pi \text{ sr}} \frac{I_\nu}{c} \, d\Omega = \frac{I_\nu}{c} \int_{4\pi \text{ sr}} d\Omega = \frac{4\pi}{c} I_\nu \quad (s1.5)$$

ii. Based on the above, show that the total energy density of black body radiation is

$$u_{\text{BB}}(T) = \int_0^\infty \frac{4\pi}{c} B_\nu \, d\nu = \frac{8\pi^5}{15} \left(\frac{kT}{hc}\right)^3 kT = aT^4 \quad (3.57)$$

*Hint:* Substitute  $x = h\nu/kT$  and note that  $\int_0^\infty \frac{x^3}{\exp(x)-1} = \pi^4/15$ .

We need to calculate

$$u_{\text{BB}}(T) = \int_0^\infty \frac{4\pi}{c} B_\nu \, d\nu \quad (s1.6)$$

$$= \frac{4\pi}{c} \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \, d\nu \quad (s1.7)$$

substituting  $x = h\nu/kT$  gives

$$= \frac{4\pi}{c} \int_0^\infty \frac{2h(kT)^3 x^3}{h^3 c^2} \frac{1}{\exp(x) - 1} \frac{kT}{h} \, dx \quad (s1.8)$$

$$= \frac{4\pi}{c} \frac{2hk^4 T^4}{h^4 c^2} \int_0^\infty \frac{x^3}{\exp(x) - 1} \quad (s1.9)$$

$$= \frac{8\pi k^4}{h^3 c^3} \frac{\pi^4}{15} \cdot T^4 \quad (s1.10)$$