Question 1: *Fourier Transforming a Differential Equation*

As shown in the lecture, one of the crucial equations to solve when looking at the emission of radiation from accelerated charges has the form

$$\nabla^2\Psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{x},t)}{\partial t^2} = -4\pi f(\mathbf{x},t) \tag{4.22}$$

which can be solved using a Green's functions Ansatz, where the Green's function $G(\mathbf{x},t;\mathbf{x}',t')$ is the solution of

$$\nabla^2 G(\mathbf{x},t;\mathbf{x}',t') - \frac{1}{c^2}\frac{\partial^2 G(\mathbf{x},t;\mathbf{x}',t')}{\partial t^2} = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{4.23}$$

The solutions of this equation can be best obtained when working in Fourier space, where the Fourier transform is defined through

$$f(\mathbf{x},\omega) = \int f(\mathbf{x},t)\mathrm{e}^{+i\omega t}\mathrm{d}t \quad \text{and} \quad G(\mathbf{x},\omega;\mathbf{x}',t') = \int G(\mathbf{x},t;\mathbf{x}',t')\mathrm{e}^{+i\omega t}\mathrm{d}t \tag{4.25}$$

and where the inverse transform is

$$f(\mathbf{x},t) = \frac{1}{2\pi}\int f(\mathbf{x},\omega)\mathrm{e}^{-i\omega t}\mathrm{d}\omega \quad \text{and} \quad G(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{2\pi}\int G(\mathbf{x},\omega;\mathbf{x}',t')\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{4.27}$$

a) By inserting the inverse Fourier transform of $G(\mathbf{x},t;\mathbf{x}',t')$, show that

$$\nabla^2 G(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{2\pi}\int \left(\nabla^2 G(\mathbf{x},\omega;\mathbf{x}',t')\right)\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{4.30}$$

Because the Fourier transform operates on t and not on t', the ∇^2 and $\int$ can be interchanged:

$$\nabla^2 G(\mathbf{x},t;\mathbf{x}',t') = \nabla^2\left(\frac{1}{2\pi}\int G(\mathbf{x},\omega;\mathbf{x}',t')\mathrm{e}^{-i\omega t}\mathrm{d}\omega\right) \tag{s1.1}$$

since ∇^2 operates on the $\mathbf{x}$ variable only, this can be written as

$$= \frac{1}{2\pi}\int \left(\nabla^2 G(\mathbf{x},\omega;\mathbf{x}',t')\right)\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{s1.2}$$

b) By inserting the inverse Fourier transform of $G(\mathbf{x},t;\mathbf{x}',t')$, show that

$$-\frac{1}{c^2}\frac{\partial^2 G(\mathbf{x},t;\mathbf{x}',t')}{\partial t^2} = \frac{1}{2\pi}\int k^2 G(\mathbf{x},\omega;\mathbf{x}',t')\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{4.33}$$

where $k = \omega/c$.

$$-\frac{1}{c^2}\frac{\partial^2 G(\mathbf{x},t;\mathbf{x}',t')}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{2\pi}\int G(\mathbf{x},\omega;\mathbf{x}',t')\mathrm{e}^{-i\omega t}\mathrm{d}\omega\right) \tag{s1.3}$$

Since the integral operates on ω only, $\int$ and ∂^2_{tt} can be exchanged:

$$= -\int \frac{1}{2\pi c^2} G(\mathbf{x}, \omega; \mathbf{x}', t')(i\omega)^2 \mathrm{e}^{-i\omega t} \mathrm{d}\omega \tag{s1.4}$$

introducing $k = \omega/c$ then gives

$$= \frac{1}{2\pi} \int k^2 G(\mathbf{x}, \omega; \mathbf{x}', t') \mathrm{e}^{-i\omega t} \mathrm{d}\omega \tag{s1.5}$$

c) Making use of one of the definitions of the δ-function,

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{i\omega t} \mathrm{d}\omega \tag{4.34}$$

(which you can only use if you are not a mathematician. . .) and of

$$\delta(t - t') = \delta(t' - t) \tag{w1.6}$$

show that

$$(-4\pi)\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') = \frac{1}{2\pi} \int (-4\pi)\delta(\mathbf{x} - \mathbf{x}') \mathrm{e}^{i\omega t'} \mathrm{e}^{-i\omega t} \mathrm{d}\omega \tag{4.36}$$

With the above

$$\delta(t - t') = \frac{1}{2\pi} \int \mathrm{e}^{i\omega(t-t')} \mathrm{d}\omega \tag{s1.6}$$

such that

$$(-4\pi)\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') = \frac{1}{2\pi} \int (-4\pi)\delta(\mathbf{x} - \mathbf{x}') \mathrm{e}^{i\omega t'} \mathrm{e}^{-i\omega t} \mathrm{d}\omega \tag{s1.7}$$