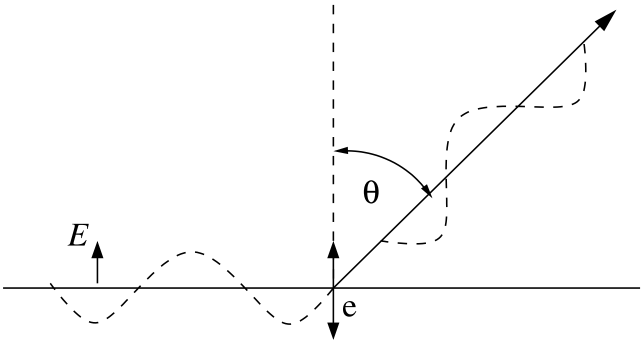
Friedrich–Alexander–Universität
Erlangen–Nürnberg

Astrophysical Radiation Processes
Sommersemester 2008
Worksheet 4

Question 1: *An Application of Larmor's Formula: Thomson Scattering and the Thomson Cross Section*



One of the simplest applications of Larmor's formula is the scattering of radiation by a free electron. In the classical approximation, this process is called Thomson scattering. It is obtained by considering a sinusoidal electromagnetic wave with E-vector

$$E(t) = E_0 \sin \omega t \tag{w1.1}$$

interacting with an electron at rest.

a) Assuming $v \ll c$, show that the force on the electron is

$$F = m_\mathrm{e} \ddot{\mathbf{r}} = eE_0 \sin \omega t \tag{w1.2}$$

and that the dipole moment of the electron is given by

$$d = -\frac{e^2 E_0}{m_\mathrm{e} \omega^2} \sin \omega t \tag{w1.3}$$

The Lorentz-force is given by

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = e\mathbf{E} \tag{2.31}$$

for $v \ll c$ (remember that for the em wave, $|E| = |B|$!). For a single electron, the dipole moment is $\mathbf{d} = e\mathbf{r}$, and therefore

$$\ddot{d} = \frac{e^2 E_0}{m_\mathrm{e}} \sin \omega t \tag{s1.1}$$

Integrating twice gives

$$d = -\frac{e^2 E_0}{m_\mathrm{e} \omega^2} \sin \omega t \tag{s1.2}$$

b) Using the results from the previous question and the dipole approximation, calculate the time averaged power emitted per unit angle and the total power emitted by the electron.
Reminder: $\langle \sin^2 \omega t \rangle = 1/2$.

Using the dipole approximation,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\ddot{d}^2}{4\pi c^3} \sin^2 \theta \qquad \text{and} \qquad P = \frac{2\ddot{d}^2}{3c^3} \tag{4.92}$$

by straightforward inserting we see

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{e^4 E_0^2}{8\pi m_\mathrm{e}^2 c^3} \sin^2 \theta \qquad \text{and} \qquad P = \frac{e^4 E_0^2}{3 m_\mathrm{e}^2 c^3} \tag{s1.3}$$

c) The *differential cross section*, $\mathrm{d}\sigma/\mathrm{d}\Omega$, is a measure how much radiation is scattered into a certain direction. It is defined through

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \langle S \rangle \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \tag{w1.4}$$

where the incident flux of radiation on the electron is given by Poynting's theorem as

$$\langle S \rangle = \frac{c}{8\pi} E_0^2 \tag{w1.5}$$

From your previous results show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{m_\mathrm{e}^2 c^4} \sin^2\theta = r_0^2 \sin^2\theta \tag{w1.6}$$

where

$$r_0 = \frac{e^2}{m_\mathrm{e} c^2} = 2.82 \times 10^{-13}\,\mathrm{cm} \tag{w1.7}$$

is called the *classical electron radius.*

Because of Eq. (s1.3),

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{e^4 E_0^2}{8\pi m_\mathrm{e}^2 c^3} \sin^2\theta \overset{!}{=} \frac{c}{8\pi} E_0^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \tag{s1.4}$$

and therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{e^4}{m_\mathrm{e}^2 c^4} \sin^2\theta = r_0^2 \sin^2\theta \tag{s1.5}$$

d) The *total cross section* for the scattering of radiation off a free electron is obtained by integrating $\mathrm{d}\sigma/\mathrm{d}\Omega$ over Ω or immediately from

$$P = \langle S \rangle \, \sigma \tag{w1.6}$$

Show that

$$\sigma = \frac{8\pi}{3} r_0^2 =: \sigma_\mathrm{T} \tag{w1.7}$$

where

$$\sigma_\mathrm{T} = 6.652 \times 10^{-25}\,\mathrm{cm}^2 \tag{w1.8}$$

is called the *Thomson cross section.*

Because of Eq. (s1.3),

$$P = \frac{e^4 E_0^2}{3 m_\mathrm{e}^2 c^3} \overset{!}{=} \langle S \rangle \, \sigma = \frac{c}{8\pi} E_0^2 \sigma_\mathrm{T} \tag{s1.6}$$

such that

$$\sigma_\mathrm{T} = \frac{8\pi}{3} \frac{e^4}{m_\mathrm{e}^2 c^4} = \frac{8\pi}{3} r_0^2 \tag{s1.7}$$

Question 2: *Parseval's Theorem*

The Fourier transform pair as used in this lecture was defined by

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{+i\omega t}\mathrm{d}t \quad \text{and} \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{w2.1}$$

Provided that $f(t)$ is a sufficiently nice function, show that Parseval's theorem holds

$$\int |f(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int |F(\omega)|\mathrm{d}\omega = \int |F(\nu)|^2\mathrm{d}\nu \tag{w2.2}$$

Reminder 1: $|f|^2 = ff^*$ where f^* is the complex conjugate of f
Reminder 2: The δ-function can be written as

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{i\omega t}\,\mathrm{d}\omega \tag{4.34}$$

The solution is pretty straightforward:

$$\int f(t)f^*(t)\mathrm{d}t = \int f(t)\int f^*(\tau)\delta(t-\tau)\mathrm{d}\tau\,\mathrm{d}t \tag{s2.1}$$

$$= \int f(t)\int f^*(\tau)\frac{1}{2\pi}\int e^{i\omega(t-\tau)}\mathrm{d}\omega\,\mathrm{d}\tau\,\mathrm{d}t \tag{s2.2}$$

$$= \frac{1}{2\pi}\iiint f(t)e^{i\omega t}f^*(t)e^{-i\omega\tau}\mathrm{d}\omega\,\mathrm{d}\tau\,\mathrm{d}t \tag{s2.3}$$

$$= \frac{1}{2\pi}\iint f(t)e^{i\omega t}\left(\int f^*(\tau)e^{-i\omega\tau}\mathrm{d}\tau\right)\,\mathrm{d}t\,\mathrm{d}\omega \tag{s2.4}$$

$$= \frac{1}{2\pi}\int\left(\int f(t)e^{i\omega t}\mathrm{d}t\right)\left(\int f^*(\tau)e^{-i\omega\tau}\mathrm{d}\tau\right)\,\mathrm{d}\omega \tag{s2.5}$$

identify Fourier transforms

$$= \frac{1}{2\pi}\int f(\omega)f^*(\omega)\mathrm{d}\omega \tag{s2.6}$$

substitute $\omega = 2\pi\nu$

$$= \int f(\nu)f^*(\nu)\mathrm{d}\nu \tag{s2.7}$$

$$\tag{s2.8}$$