Astrophysical Radiation Processes
Sommersemester 2008
Worksheet 5

Question 1: *Fourier Transform of Time Derivatives*

The Fourier transform pair was defined in the lectures via

$$f(\omega) = \int f(t)\mathrm{e}^{i\omega t}\mathrm{d}t \quad \Longleftrightarrow \quad f(t) = \frac{1}{2\pi}\int d(\omega)\mathrm{e}^{-i\omega t}\mathrm{d}\omega \tag{w1.1}$$

Show that the Fourier transform of the time derivative of $f(t)$, $\dot{f}(\omega)$, can be calculated from the very useful equation

$$\dot{f}(\omega) = -i\omega f(\omega) \tag{w1.2}$$

and therefore

$$\ddot{f}(\omega) = \omega^2 f(\omega) \tag{w1.3}$$

Hint: Partial integration!
Hint 2: A necessary condition for a Fourier transform to exist is $\lim_{t\to\pm\infty} f(t) = 0$

The solution is best found through straightforward calculation:

$$\dot{f}(\omega) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{i\omega t}\,\mathrm{d}t \tag{s1.1}$$

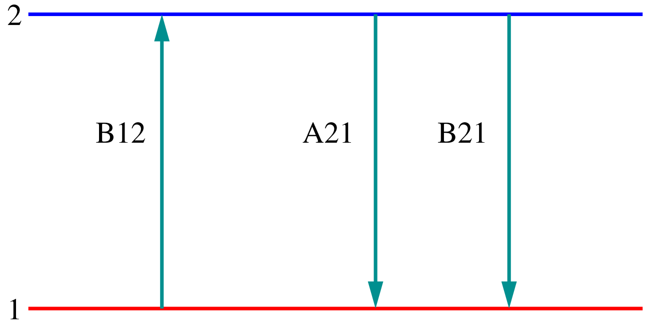
partial integration gives

$$= \left[f(t)\mathrm{e}^{i\omega t}\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t)(i\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}t \tag{s1.2}$$

because $\lim_{t\to\pm\infty} f(t) = 0$ this gives

$$= -i\omega f(\omega) \tag{s1.3}$$

Question 2: *Self-Absorption and the Principle of Detailed Balance*



The emission and absorption of a 2-level system can be described with the Einstein-coefficients, where A_{21} is the transition probability per unit time for spontaneous emission, and where $B_{12}I_\nu$ and $B_{21}I_\nu$ are the transition probabilities for absorption and for stimulated emission per unit time.

As shown in the lectures, these coefficients are not independent of each other, but obey the Einstein relations:

$$g_1 B_{12} = g_2 B_{21} \quad \text{and} \quad A_{21} = \frac{2h\nu_{21}^3}{c^2} B_{21} \tag{w2.1}$$

a) Let n_1 and n_2 be the population densities of level 1 and 2. Convince yourself that the equation of radiative transfer,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu \tag{w2.2}$$

can be written in terms of Einstein coefficients as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = \frac{h\nu_{21} n_2 A_{21}}{4\pi} - n_1 B_{12} h\nu_{21} I_\nu + n_2 B_{21} h\nu_{21} I_\nu \tag{w2.3}$$

In principle, this is fairly obvious if one remembers that an emitted photon carries an energy $h\nu$. The second difficulty is due to the factor of 4π related to A_{21}. This factor is due to the fact that emission is isotropic, i.e., only a factor $1/4\pi$ is emitted into the $\mathrm{d}\Omega$ along the line of sight.

b) Show from the above equation that

$$j_\nu = \frac{h\nu_{21} n_2 A_{21}}{4\pi} \tag{w2.1}$$

Trivial

c) Noting that in thermodynamic equilibrium

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left(-\frac{h\nu_{21}}{kT}\right) \tag{w2.1}$$

show that

$$\alpha_\nu = h\nu_{21} n_1 B_{12} \left(1 - \exp\left(-\frac{h\nu_{21}}{kT}\right)\right) \tag{w2.2}$$

For the absorptivity, first identify

$$\alpha_\nu = h\nu_{21}(n_1 B_{12} - n_2 B_{21}) = h\nu_{21} n_1 B_{12} \left(1 - \frac{n_2 g_1}{n_1 g_2}\right) \tag{s2.1}$$

Because of Boltzmann,

$$\frac{n_2}{n_1}\frac{g_1}{g_2} = \exp\left(-\frac{h\nu_{21}}{kT}\right) \tag{s2.2}$$

and inserting this into Eq. (s2.1) gives the desired result.