Friedrich-Alexander-Universität
Erlangen-Nürnberg

Astrophysical Radiation Processes
Sommersemester 2008
Worksheet 7

Question 1: *Lifetime of a synchrotron radiation emitting electron*

In the lecture it was shown that the radiation emitted from a relativistic electron with energy $E = \gamma m_\mathrm{e} c^2$ is given by

$$\langle P_\mathrm{em} \rangle = \frac{4}{3} \gamma^2 c \sigma_\mathrm{T} U_\mathrm{B} \tag{6.10}$$

A consequence of this equation is that electrons emitting synchrotron radiation loose their energy.

a) Show that the energy of a synchrotron radiation emitting electron at time t is given by

$$E(t) = \frac{E_0}{1 + t/t_{1/2}} \quad \text{with} \quad t_{1/2} = \frac{3 m_\mathrm{e} c^3}{4 \sigma_\mathrm{T} U_\mathrm{B}} \frac{1}{E_0} \tag{w1.1}$$

where $E_0 = E(t = 0)$ is the electron's initial energy.

We write Eq. (6.10) as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\xi E^2 \quad \text{with} \quad \xi = \frac{4}{3} \frac{1}{m_\mathrm{e}^2 c^3} \sigma_\mathrm{T} U_\mathrm{B} \tag{s1.1}$$

Separation of variables gives

$$\int_{E_0}^{E(t)} \frac{\mathrm{d}E}{E^2} = -\xi \int_0^t \mathrm{d}t \quad \Longrightarrow \quad \frac{1}{E_0} - \frac{1}{E(t)} = -\xi t \tag{s1.2}$$

Solving for $E(t)$ then gives the desired result.

b) Show that the life time $t_{1/2}$ can also be obtained from

$$t_{1/2} = \frac{E_0}{\langle P \rangle} \tag{w1.3}$$

Insert $E_0 = \gamma m_\mathrm{e} c^2$ and perform some trivial algebra.

c) *Do this subquestion at home.* Show that for typical conditions in supernova remnants such as the Crab ($E_0 = 1\,\mathrm{GeV}$ and $B = 10^{-4}\,\mathrm{G}$),

$$t_{1/2} = 4 \times 10^6\,\mathrm{yr} \left(\frac{B}{10^{-4}\,\mathrm{G}} \right)^{-2} \left(\frac{E_0}{1\,\mathrm{GeV}} \right)^{-1} \tag{w1.3}$$

The Crab nebula has a size of only a few parsecs. The electrons producing the synchrotron radiation from the nebula can therefore originate from the central pulsar. The situation is different in jets from active galactic nuclei. These jets can have lengths of several 100 kpc and the typical B-fields are a few 10^{-4} G. Since the jets are significantly longer than $c t_{1/2}$, one has to infer that there is a source of energy within the jets. The physical process responsible for this source, however, is not yet known.

Hint: A good collection of cgs constants needed in this exercise can be found at `http://www.astro.wisc.edu/~dolan/constants/calc.html`.

In exercises such as this one it is generally best to calculate τ first for $E = 1\,\mathrm{erg}$ and $B = 1\,\mathrm{G}$ and then scale to the "typical" numbers later. Normally this avoids mistakes due to unit conversion… Therefore, using $1\,\mathrm{GeV} = 1.6 \times 10^{-3}\,\mathrm{erg}$,

$$\begin{aligned}\tau = \frac{3m_\mathrm{e}^2c^3}{4\sigma_\mathrm{T}U_\mathrm{B}E_0} &= \frac{24\pi^2m_\mathrm{e}^2c^3}{4\sigma_\mathrm{T}}B^{-2}E_0^{-1}\\ &= 1986\,\mathrm{s}\left(\frac{B}{1\,\mathrm{G}}\right)^{-2}\left(\frac{E_0}{1\,\mathrm{erg}}\right)^{-1} = 1986\,\mathrm{s}\left(\frac{10^{-4}B}{10^{-4}\,\mathrm{G}}\right)^{-2}\left(\frac{E_0}{625\,\mathrm{GeV}}\right)^{-1}\\ &= 1.2\times10^{14}\,\mathrm{s}\left(\frac{B}{10^{-4}\,\mathrm{G}}\right)^{-2}\left(\frac{E_0}{1\,\mathrm{GeV}}\right)^{-1} = 4\times10^{6}\,\mathrm{yr}\left(\frac{B}{10^{-4}\,\mathrm{G}}\right)^{-2}\left(\frac{E_0}{1\,\mathrm{GeV}}\right)^{-1} \qquad \text{(s1.3)}\end{aligned}$$

Question 2: *Power law slope for Compton scattering*

a) A slab of thickness ℓ and electron number density n (measured in units of electrons cm^{-3}) is irradiated by light with initial intensity N_0 (where N is the number of photons per second and square-cm). Convince yourself that due to scattering, the decrease in photon number over infinitesimal distance dx is given by

$$\frac{\mathrm{d}N}{N} = -n\sigma\mathrm{d}x \qquad \text{(w2.1)}$$

where σ is the Thomson cross section. Use Eq. w2.1 to show that the number of photons emerging on the other side of the slab in the original direction of the photons is

$$N(\ell) = N_0\exp(-\tau) \qquad \text{(w2.2)}$$

where $\tau = n\sigma\ell$. (*Hint:* The decrease in photon flux within an infinitely thick slab is $\mathrm{d}N = -n\sigma N\mathrm{d}x$.)

For an infinitely thin slab, the area density of electrons is given by $n \cdot dx$ (units: electrons per square-cm). For the purposes of scattering, each electron can be seen to have an area σ, such that the area $n\sigma\mathrm{d}x$ is can be considered covered by the electrons. Once a photon hits this area, it is scattered out of the line of sight, and therefore the decrease in photon flux is

$$\mathrm{d}N = -n\sigma N\mathrm{d}x \qquad \text{(s2.1)}$$

Writing this equation as

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}x} = -n\sigma \qquad \text{(s2.2)}$$

and integrating over $\mathrm{d}x$ then gives

$$\int_0^\ell \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}x}\mathrm{d}x = -\int_0^\ell n\sigma\mathrm{d}x \quad\longleftrightarrow\quad \log\left(\frac{N(\ell)}{N_0}\right) = -n\sigma\ell \qquad \text{(s2.3)}$$

and therefore

$$N(\ell) = N_0\exp(-n\sigma\ell) = N_0\exp(-\tau) \qquad \text{(s2.4)}$$

b) Using Eq. (w2.2), convince yourself that the probability of a photon to travel at least an optical depth τ is

$$p(\tau) = \exp(-\tau) \qquad \text{(w2.5)}$$

and that the mean optical depth traveled before the photon scatters, $\langle\tau\rangle = 1$.

(*Hint:* if x has a probability density distribution $\propto p(x)$, then its mean is $\langle x\rangle = \int xp(x)\mathrm{d}x/\int p(x)\mathrm{d}x$.)

The mean optical depth traveled is given by

$$\langle\tau\rangle = \frac{\int_0^\infty \tau \exp(-\tau)\mathrm{d}\tau}{\int_0^\infty \exp(-\tau)\mathrm{d}\tau} \tag{s2.5}$$

The denominator integrates to 1 such that

$$\langle\tau\rangle = \int_0^\infty \tau \exp(-\tau) d\tau = [\exp(-\tau)\tau]_0^\infty + \int_0^\infty \exp(-\tau)\mathrm{d}\tau = 1 \tag{s2.6}$$

where partial integration was used to solve the integral.

c) Use the result from the previous question to show that the mean physical distance traveled in the slab, the *mean free path* l, is

$$l = \frac{1}{n\sigma} \tag{w2.7}$$

The mean optical depth traveled is $\langle\tau\rangle = n\sigma\ell = 1$, and therefore the mean distance traveled between scatters is

$$\ell = \frac{1}{n\sigma} \tag{s2.7}$$

d) Show that for *small* τ the probability of a photon undergoing k scatterings before escaping the medium is approximately

$$p_k(\tau) \sim \tau^k \tag{w2.8}$$

(*Hint:* First look at the probability that the photon escapes after *one* scattering and then use induction.)

The probability that the photon escapes after one scattering is

$$p_1(\tau) = 1 - \exp(-\tau) \sim \tau \tag{s2.8}$$

for τ small. The desired answer then follows by induction.

e) For Compton scattering and a seed photon energy $E_\mathrm{s} \ll kT$, as shown in the lecture the amplification factor is

$$A \sim \frac{4kT}{mc^2} \tag{w2.9}$$

Show that after k scatterings the energy of the seed photon, E_k, is approximately

$$E_k \sim E_\mathrm{s} A^k \tag{w2.10}$$

(*Hint:* proof by induction.)

The energy of the photon after one scattering is

$$E_1 = E_\mathrm{s} A \tag{s2.9}$$

and by induction, its energy after k scatterings is

$$E_k = E_{k-1} A = E_\mathrm{s} A^k \tag{s2.10}$$

f) Using Eqs. w2.8 and w2.10, show that the emergent intensity at energy E_k is a power law

$$N(E_k) = N(E_\mathrm{s}) \left(\frac{E_k}{E_\mathrm{s}}\right)^{-\alpha} \quad \text{where} \quad \alpha = -\frac{\ln\tau}{\ln A} \tag{w2.11}$$

(*Hint:* estimate the intensity emerging at energy E_k after k scatterings and then play with logarithms.)

The intensity emerging at energy E_k is approximately proportional to $p_k(\tau)$ since to first order only photons upscattered to E_k will contribute to the emerging spectrum. Therefore

$$N(E_k) = N(E_\mathrm{s}) p_k(\tau) = N(E_\mathrm{s})\tau^k \tag{s2.11}$$

But because of $E_k = E_\mathrm{s} A^k$,

$$k = \ln(E_k/E_\mathrm{s})/\ln A \tag{s2.12}$$

and therefore

$$\begin{aligned}\tau^k &= \tau^{\ln(E_k/E_\mathrm{s})/\ln A} = (\exp(\ln\tau))^{\ln(E_k/E_\mathrm{s})/\ln A} \\ &= \exp\left(\ln(E_k/E_\mathrm{s})\ln\tau/\ln A\right) = \left(\frac{E_k}{E_\mathrm{s}}\right)^{\ln\tau/\ln A} = \left(\frac{E_k}{E_\mathrm{s}}\right)^{-\alpha}\end{aligned} \tag{s2.13}$$

with $\alpha = -\ln\tau/\ln A$.

Question 3: *Synchrotron self-Compton Emission and the Compton catastrophe*

Since synchrotron radiation is produced by highly energetic electrons, it is not unlikely that synchrotron photons are Compton scattered subsequent to their emission. This process is called Synchrotron self-Compton emission, and normally abbreviated SSC.

In the lectures it was shown that the power gained in Compton scattering is

$$P_\mathrm{compt} = \frac{4}{3}\sigma_\mathrm{T} c\gamma^2\beta^2 U_\mathrm{rad} \tag{w3.1}$$

while the synchrotron emissivity of a plasma was

$$P_\mathrm{synch} = \frac{4}{3}\sigma_\mathrm{T} c\gamma^2\beta^2 U_\mathrm{B} \tag{w3.2}$$

a) The *Compton catastrophe* occurs when synchrotron photons are so violently Compton upscattered that the scattering electrons loose all of their energy, which will happen when $P_\mathrm{compt} > P_\mathrm{synch}$. Show that for a source of angular size θ at a distance d the Compton catastrophe will happen if the ratio

$$\frac{L_\mathrm{compt}}{L_\mathrm{synch}} = \frac{U_\mathrm{rad}}{U_\mathrm{B}} = \frac{8\pi F}{B^2\theta^2 c} \tag{w3.3}$$

is greater than 1. Here, F is the observed synchrotron flux from the source and $U_\mathrm{rad} = L_\mathrm{synch}/4\pi r^2 c$ is the energy density within the synchrotron source, which has a radius r. (*Hint:* $F = L/4\pi d^2$ and $\theta = r/d$ in the small angle approximation)

The power emitted from a source is proportional to its luminosity, such that

$$\frac{L_\mathrm{compt}}{L_\mathrm{synch}} = \frac{P_\mathrm{compt}}{P_\mathrm{synch}} = \frac{U_\mathrm{rad}}{U_\mathrm{B}} \tag{s3.1}$$

Since

$$U_\mathrm{rad} = \frac{L_\mathrm{synch}}{4\pi r^2 c} = \frac{4\pi r^2 F}{4\pi r^2 c\theta^2} = \frac{F}{\theta^2 c} \tag{s3.2}$$

and

$$U_{\rm B} = \frac{B^2}{8\pi} \tag{s3.3}$$

one finds

$$\frac{L_{\rm compt}}{L_{\rm synch}} = \frac{8\pi F}{\theta^2 B^2 c} \tag{s3.4}$$

b) The *brightness temperature*, T, is used in radio astronomy to describe the intensity, I, of a radio source. It is defined by

$$I = \frac{2\nu^2 kT}{c^2} \tag{w3.5}$$

In addition, if one observes an unresolved radio source of angular size θ a flux $F = \theta^2 I$ is measured. Close to the Compton limit, the source must have a high surface brightness such that the source starts to be self-absorbed. The total flux from the source is therefore approximately $F = F(\nu_a)\nu_a$, where $F(\nu_a)$ is the source flux at the frequency ν_a, where self-absorption sets in. This frequency is approximately given by

$$\nu_a = \frac{3}{2}\nu_{\rm c} \tag{w3.6}$$

where $\nu_{\rm c} = \omega_{\rm c}/2\pi$ with $\omega_{\rm c} = \gamma^2\omega_{\rm L}$ being the characteristic frequency of synchrotron radiation (see lecture, Eq. 6.15). Using this information, show that

$$\frac{L_{\rm compt}}{L_{\rm synch}} \sim 16\pi \left(\frac{3e}{4\pi m_{\rm e}^3 c^5}\right)^2 \frac{(kT)^5}{c^3}\nu_a \sim \left(\frac{T}{10^{12.1}\,{\rm K}}\right)^5 \frac{\nu_a}{100\,{\rm GHz}} \tag{w3.7}$$

Therefore, when observing opaque (=self-absorbed) sources at 100 GHz, we do not expect to see sources with brightness temperatures above 10^{12} K. This result was first noticed by Kellermann and Pauliny-Toth in 1969. If a source is found to exceed the Compton limit, this is typically due to variability and/or relativistic beaming.

Hint: Since we are observing at the frequency where self-absorption sets in, the energy of electrons radiating at that frequency is $E = kT$ where T is the observed brightness temperature.

The question already gives the equations required to find the maximum observable brightness temperature. First of all

$$\frac{F_a}{\theta^2} = \frac{2\nu_a^2 kT}{c^2} \tag{s3.5}$$

With the information given above, ν_a is found to be

$$\nu_a \sim \frac{3eB}{4\pi m_{\rm e} c}\left(\frac{kT}{m_{\rm e}c^2}\right)^2 \tag{s3.6}$$

where we set $E = kT$. Insert everything into Eq. (s3.4) then gives the desired result.