1 Ellipse-geometry

1.1 Parameterization

- **Functional characterization**: (a: semi major axis, $b \leq a$: semi minor axis)

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \;=\; 1 \qquad \Longleftrightarrow \qquad y(x) \;=\; \frac{b}{a} \cdot \left(\pm\sqrt{a^2 - x^2}\right) \tag{1}$$

- **Parameterization in cartesian coordinates**, which follows directly from Eq. (1):

$$\begin{pmatrix} x \\ y \end{pmatrix} \;=\; \begin{pmatrix} a \cdot \cos t \\ b \cdot \sin t \end{pmatrix} \quad \text{with } \; 0 \leq t < 2\pi \tag{2}$$

  - The origin (0, 0) is the center of the ellipse and the auxilliary circle with radius a.
  - The focal points are located at $(\pm a \cdot e,\ 0)$ with the eccentricity $e = \sqrt{a^2 - b^2}/a$.

- **Parameterization in polar coordinates**: (p: parameter, $0 \leq \epsilon < 1$: eccentricity)

$$r(\varphi) \;=\; \frac{p}{1 + e\cos\varphi} \tag{3}$$

  - The origin (0, 0) is the right focal point of the ellipse.
  - The major axis is given by $2a = r(0) - r(\pi)$, thus $a = p/(1 - e^2)$, the center is therefore at $\big(-pe/(1 - e^2),\ 0\big)$.
  - $\varphi = 0$ corresponds to the periapsis (the point closest to the focal point; which is also called perigee/perihelion/periastron in case of an orbit around the Earth/sun/star).

The relation between t and φ of the parameterizations in Eqs. (2) and (3) is the following:

$$\tan\left(\frac{t}{2}\right) \;=\; \sqrt{\frac{1-e}{1+e}} \cdot \tan\left(\frac{\varphi}{2}\right) \tag{4}$$

1.2 Area of an elliptic sector

As an ellipse is a circle with radius a scaled by a factor b/a in y-direction (Eq. 1), the area of an elliptic sector PFS (Fig. **??**) is just this fraction of the area PFQ in the auxiliary circle.

$$\begin{aligned} A_{\text{PFS}} \;&=\; \frac{b}{a} \cdot \left(\frac{t}{2\pi} \cdot \pi a^2 - \frac{1}{2} \cdot ae \cdot a\sin t\right) \\ &=\; \frac{1}{2}\,(t - e\sin t) \cdot a\,b \end{aligned} \tag{5}$$

The area of the full ellipse ($t = 2\pi$) is then, of course,

$$A_{\text{ellipse}} \;=\; \pi\, a\, b \tag{6}$$

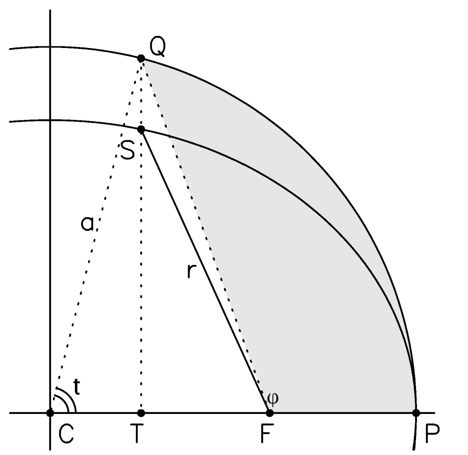

Figure 1: Ellipse and auxilliary circle.

2 Orbit in a binary system

2.1 Parameterization of the orbit

The geometry of an elliptical orbit is usually parameterized by the semi major axis a and the eccentricity e (see § 1.1).
The orientation of the orbital plane (P1 in Fig. **??**) relative to a reference plane P2 (the sky or the ecliptic) can be specified by 3 parameters:

- the inclination i,
  which is the angle between these planes ($i = 90°$ systems are seen edge-on.)
- the argument of periapsis, ω,
  which is the angle in the orbital plane P1 from the ascending node to the periapsis
- the longitude of the acending node, Ω,
  which is the angle in the plane P2 from a 0-direction to the ascending node.

(The nodes are the intersection points of the orbit with the reference plane P2. If the direction of the orbit is known, ascending and descending node can be distinguished.)

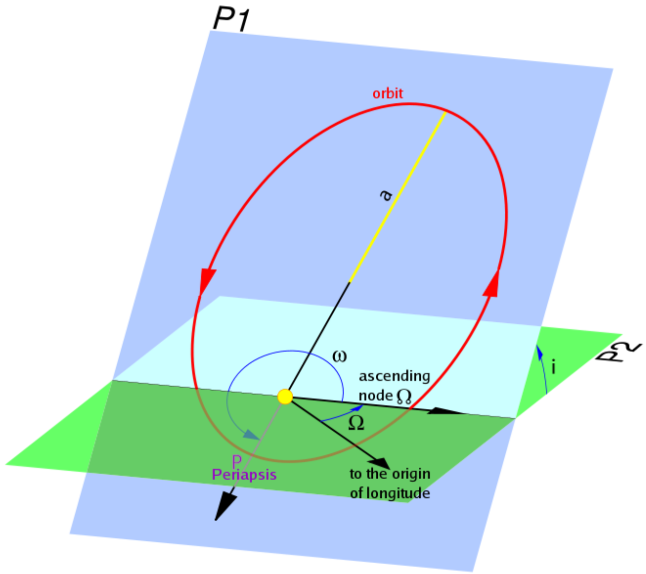

Figure 2: Orbital elements (after http://commons.wikimedia.org/wiki/Image:Orbital_elements.svg)

Derived angle

The longitude of the periapsis $\varpi = \Omega + \omega$ is composed of angles in different planes (unless $i = 0$) – the longitude of the acending node Ω in P2 and the argument of periapsis ω in P1.

2.2 Parameterization of the position on the orbit

The position on the curve can be specified by a single parameter. The following conventions are in use:

◇ the true anomaly ν, which is just the φ of the polar parameterization, Eq. (3)

◇ the eccentric anomaly E, which is just the t of the cartesian parameterization, Eq. (2)

◇ the true longitude $l = \varpi + \nu$ (which is again composed of angels in different planes)

2.3 Orbital dynamics

Mean angles and epochs

Although an elliptical orbit is, according to Kepler's law, not performed with constant angular velocity, the following "mean" quantities are defined to evolve uniformly in time:

◇ the mean anomaly $M = 2\pi \cdot (t - T_\omega)/P_{\rm orb}$, with an epoch T_ω of periapsis ($\nu = 0$) and the period $P_{\rm orb}$

◇ the mean longitude $L = \varpi + M$ (which is again composed of angles in diffrerent planes)

For elliptic orbits, the time $T_{\pi/2}$ "of mean longitude 90°" (though $\omega + M = \frac{\pi}{2}$ is meant instead of $\varpi + M = \frac{\pi}{2}$)

$$T_{\pi/2} \;=\; T_\omega \;+\; \left(\frac{\pi}{2} - \omega\right) \cdot \frac{P_{\rm orb}}{2\pi} \tag{7}$$

is often given as refrence epoch, as it does not depend on ω as strong as the time T_ω of periapsis, of course. A superior conjunction requires, however, that the true (instead of the mean) anomaly equals $\nu = \frac{\pi}{2} - \omega$.

Time evolution of the true anomaly

From Kepler's law, the true anomaly ν has to evolve such that the area of the elliptic sector grows linarly with M (time). The parameter t of Eq. (5) is the eccentric anomaly E – which is related to the true anomaly $\nu \equiv \varphi$ by Eq. (4). The area of an elliptic sector, Eq. (5), relates to the total area, Eq. (6), as M to 2π:

$$\frac{E - e\sin E}{2\pi} \;=\; \frac{M}{2\pi} \;=\; \frac{t - T_\omega}{P_\mathrm{orb}} \tag{8}$$

The time T_conj of superior conjunction can thus be calculated from $\nu = \frac{\pi}{2} - \omega$:

$$T_\mathrm{conj} \;=\; T_\omega \;+\; \Big(E + e\sin E\Big) \cdot \frac{P_\mathrm{orb}}{2\pi}, \qquad \text{where} \quad E \;=\; 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\,\tan\left(\frac{\pi}{4} - \frac{\omega}{2}\right)\right) \tag{9}$$

Series expansion in e gives now:

$$T_\mathrm{conj} \;=\; \underbrace{T_\omega + \left(\frac{\pi}{2} - \omega\right) \cdot \frac{P_\mathrm{orb}}{2\pi}}_{=\,T_{\pi/2}} \;-\; e \cdot \frac{P_\mathrm{orb} \cdot \cos\omega}{\pi} \;+\; \mathrm{O}(e^2) \tag{10}$$

It is, however, noted, that the time of superior conjunction does not have to coincide with the mid-ecclipse time of an eclip high-inclination system.