Friedrich-Alexander-Universität
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Active Galactic Nuclei
Sommersemester 2010
Übungsaufgaben 1b – Solution
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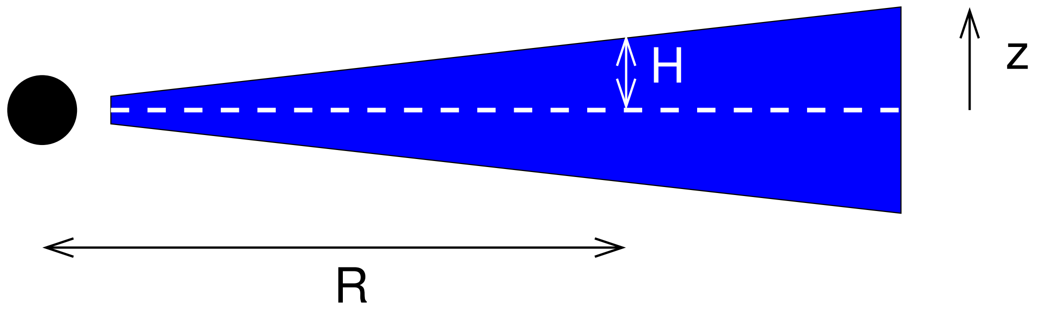
Question 1: *Accretion Disks*



In this exercise we will be looking in more detail at the structure of thin accretion disks, i.e., at disks for which the height of the disk, H, scales with distance R from the central compact object such that $H \ll R$.

a) **Accretion disks are strongly supersonic**. In order to get an order of magnitude feeling for the condition within the accretion disk, we will first take a look at the vertical structure of a disk, i.e., at the z-direction. Since thin disks are gas pressure supported, we will ignore radiation pressure. For a stationary system, this means that the gravitational force in z-direction (which you should estimate for the limit $z \ll R$) is balanced by the force due to the pressure gradient in that direction. Convince yourself that the force due to a pressure gradient scales as

$$\frac{1}{\rho}\frac{\partial P}{\partial z} \approx \frac{1}{\rho_c}\frac{P_c}{H} \tag{1.1}$$

where P_c is some characteristic pressure and ρ_c some characteristic density. Using these assumptions, show that accretion disks must be strongly supersonic

(*Hint:* the speed of sound is $c_s = \sqrt{P/\rho}$, the Kepler speed is $v_\phi = \sqrt{GM/R}$).

Solution:

$$\frac{GM}{R^2}\frac{H}{R} = \frac{1}{\rho}\left|\frac{\partial P}{\partial z}\right| \sim \frac{P_c}{\rho_c H} = \frac{c_s^2}{H} \tag{s1.1}$$

But this means that

$$c_s^2 = \frac{GM}{R}\frac{H^2}{R^2} = v_\phi^2 \cdot \frac{H^2}{R^2} \tag{s1.2}$$

and since $H/R \ll 1$ $v_\phi \gg c_s$.

b) **The vertical density decays exponentially**

By calculating the gravitational potential energy gained when moving a particle slightly out of the midplane of the disk, convince yourself by analogy with the Earth's atmosphere that the vertical density gradient in the accretion disk must scale as

$$n_R(z) \propto \exp\left(-\frac{z}{H}\right) \tag{1.2}$$

where $n_R(z)$ is the particle density at radius R and height z (you can assume the disk to be isothermal).

Solution: One way to come to this result is to look at what happens with the gravitational potential energy when a test particle is moved away from the $z = 0$ plane. The change in energy is

$$\Delta E = \frac{GM}{R + \Delta R} - \frac{GM}{R} = \frac{GM}{R} \frac{1}{1 + \frac{\Delta R}{R}} - \frac{GM}{R} = \frac{GM}{R} \left(\frac{1}{1 + \frac{\Delta R}{R}} - 1 \right) = \frac{GM}{R} \frac{\Delta R}{R} \tag{s1.3}$$

For for small deviations from the $z = 0$ plane $\Delta R \sim z$, this means that the change in Energy is

$$\Delta E \propto z \tag{s1.4}$$

which looks exactly like the good old $E = mgh$ and therefore everything one ever learned about isothermal atmospheres kicks in, including that it has an exponential density profile...

(an alternative way to see this is to do a Taylor on $1/\sqrt{R^2 + z^2}$ for $z \ll R$)

c) **Gas particles move on quasi-spherical orbits** In the lectures it was claimed that the gas motion in the accretion disk is primarily on circular orbits, or, in other words, that the radial velocity of gas in the disk, v_R, is significantly smaller than the Kepler speed, v_ϕ. Using the same approximations as above, show that the radial acceleration due to a radial gradient in gas pressure is negligable compared to the acceleration due to gravitation, and that therefore $v_R \ll v_\phi$.

Solution:

$$\frac{1}{\rho} \frac{\partial P}{\partial R} \sim \frac{P_c}{\rho_c R} \sim \frac{c_s^2}{R} \sim \frac{GM}{R^2} \frac{H^2}{R^2} \ll \frac{GM}{R^2} \tag{s1.5}$$

d) **Mass Conservation**

In a stationary disk, the mass flow towards the black hole through the disk, $\dot{M}$, must be conserved. Assuming you know v_R and using the *surface density*

$$\Sigma(R) = \int n_R(z)\, \mathrm{d}z \tag{1.3}$$

write down an equation connecting $\dot{M}$ and $\Sigma(R)$ (this can be used to calculate the normalization constant in Eq. (1.2), although you need not do so here).

Solution:

$$\dot{M} = -2\pi R \cdot \Sigma \cdot v_R \tag{s1.6}$$

(since v_R is negative, the minus-sign is required to ensure that $\dot{M}$ is positive).

e) **Angular Momentum Transport**

Since the disk is rotating differentially, we need to transport angular momentum away. This is somewhat tricky...

i) Calculate the specific angular momentum, i.e., the angular momentum per unit mass, for a ring of the accretion disk that rotates with the Kepler speed, and show that the ring material needs to loose angular momentum in order to move towards the black hole.

Solution:

$$\mathcal{L} = R \cdot v_\phi = R\sqrt{\frac{GM}{R}} \propto R^{1/2} \tag{s1.7}$$

therefore outer rings have higher angular momentum, i.e., to move inwards, the ring needs to loose angular momentum.

ii) To get rid of the angular momentum, viscous forces are invoked. The definition of the *coefficient of kinematic viscosity*, ν, is such that the force per unit length of between two media moving with different angular speeds, Ω, is

$$\mathcal{F} = \nu\Sigma \cdot R\frac{d\Omega}{dR} \tag{1.4}$$

For Kepler motion,

$$\Omega = \sqrt{\frac{GM}{R^3}} \tag{1.5}$$

Solution: The torque per unit length is given by "force times length", i.e.,

$$G(R) = 2\pi R\mathcal{F} \cdot R = \nu\Sigma 2\pi R^3 \left(\frac{d\Omega}{dR}\right) \tag{s1.8}$$

iii) So far, you have calculated the torque between *two* rings. However, the disk consists of *many* rings, and therefore the total torque available for balancing the change in angular momentum is the net torque $\frac{dG}{dR}$ only. Use this *Ansatz* to show that

$$\nu\Sigma = \frac{\dot{M}}{3\pi} + \frac{\text{const.}}{R^{1/2}} \tag{1.6}$$

(*Note:* The change in angular momentum per unit time to be balanced by the torque is $\dot{M}d\mathcal{L}/dR$)

Solution: Balancing the torque and the change in angular momentum gives

$$\dot{M}\frac{d(R^2\Omega)}{dR} = -\frac{d}{dR}\left(\nu\Sigma 2\pi R^3\frac{d\Omega}{dR}\right) \tag{s1.9}$$

insert Ω and do some algebra

$$\dot{M}\frac{d}{dR}\left(R^2\sqrt{\frac{GM}{R^3}}\right) = \dot{M}\frac{d}{dR}\sqrt{GMR} = \frac{d}{dR}\left(\nu\Sigma 3\pi R^3\frac{(GM)^{1/2}}{R^{5/2}}\right) = \frac{d}{dR}\left(3\pi\nu\Sigma(GMR)^{1/2}\right) \tag{s1.10}$$

which means that

$$\nu\Sigma R^{1/2} = \frac{\dot{M}}{3\pi}R^{1/2} + \text{const.} \tag{s1.11}$$

iv) The constant in Equation (1.6) depends on the so-called inner boundary condition. For a black hole, one often makes the assumption that no torque is acting at the inner edge of the accretion disk, at radius R_* (for black holes, $R_* = 6GM/c^2$, but this is not important here), i.e., that $G(R_*) = 0$. Show that this means that

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{1.7}$$

(for disks around a solid body, e.g., a young star or a neutron star, the inner boundary condition is different, since the rotational speeds of the central object and of the disk must match at the surface of the object ("no slip boundary condition"). This usually leads to a "bounday layer" where significant amounts of energy are dissipated).

Solution: With

$$G(R_*) = -\frac{3}{2}\nu\Sigma 2\pi\sqrt{GMR_*} = -\frac{3}{2}\left(\frac{\dot{M}}{3\pi} + \frac{\text{const.}}{R_*^{1/2}}\right)2\pi\sqrt{GMR_*} \stackrel{!}{=} 0 \tag{s1.12}$$

such that

$$\text{const.} = -\frac{R_*^{1/2}\dot{M}}{3\pi} \tag{s1.13}$$

and therefore

$$\nu\Sigma = \frac{\dot{M}}{3\pi} + \frac{\text{const.}}{R^{1/2}} = \frac{\dot{M}}{3\pi} - \frac{\dot{M}}{3\pi}\frac{R_*^{1/2}}{R^{1/2}} = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{s1.14}$$

(Note that in conjunction with Eq. (s1.6) this equation can be used to evaluate v_R!)

f) **Temperature Profile**

The energy dissipated per unit area can be calculated from

$$D(R) = \nu\Sigma\left(R\frac{d\Omega}{dR}\right)^2 \tag{1.8}$$

Assuming the disk is optically thick, use the Stefan-Boltzmann law and your previous results to determine the temperature profile of the accretion disk.

(*Note:* Do not forget that the disk has two sides...)

Solution:

$$D(R) = \nu\Sigma\left(R\frac{d\Omega}{dR}\right)^2 = \frac{\nu\Sigma}{2}\left(-\frac{3}{2}\sqrt{\frac{GM}{R^3}}\right)^2 = \frac{9}{8}\nu\Sigma\frac{GM}{R^3} = \frac{3GM\dot{M}}{8\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right] \tag{s1.15}$$

In the simplest disks, the dissipated energy has to be radiated away locally, such that

$$\sigma_{\mathrm{SB}}T(R)^4 = D(R) \tag{s1.16}$$

and therefore

$$T(R) = \left\{\frac{3GM\dot{M}}{8\pi R^3\sigma_{\mathrm{SB}}}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]\right\}^{1/4} \tag{s1.17}$$

g) **Total disk luminosity**

Assuming the disk's outer radius is at infinity, calculate the total luminosity of the accretion disk. Compare your result with the total available energy.

Solution: The disk's bolometric luminosity is given by (note two sides!)

$$L = 2\int_{R_*}^{\infty} 2\pi R D(R) dR = \frac{3GM\dot{M}}{2}\int_{R_*}^{\infty}\frac{1}{R^2}\left(1 - \left(\frac{R_*}{R}\right)^{1/2}\right)dR$$

$$= \frac{3GM\dot{M}}{2}\left[-R^{-1} + \frac{2R_*^{1/2}}{3}R^{-3/2}\right]_{R_*}^{\infty} = \frac{GM\dot{M}}{2R_*} \tag{s1.18}$$

This is half the potential energy available for radiating way. The other half gets “advected” into the black hole, or, in the case of accretion onto a solid body, is radiated away by the boundary layer between the solid body and the disk proper (this is the case, e.g., in accretion onto neutron stars, which therefore are brighter than black hole systems).