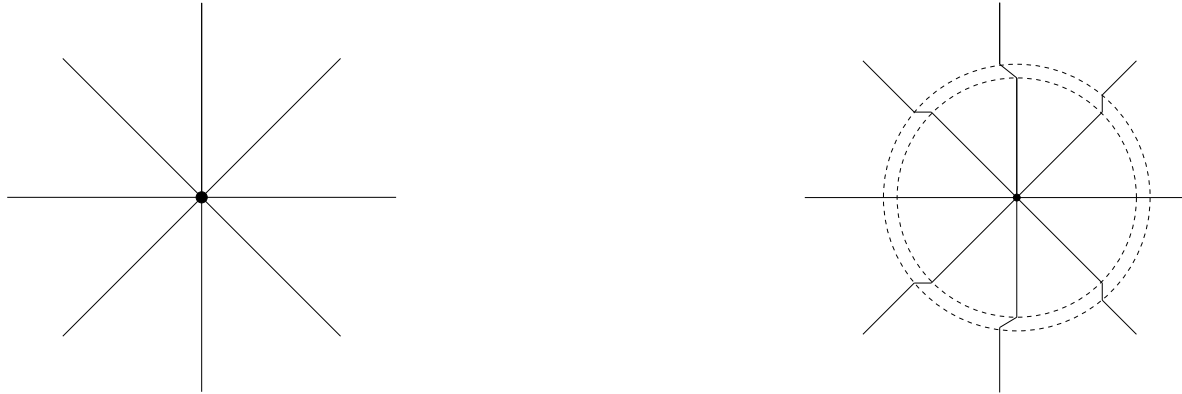


Larmor's Formula

One of the most important results of classical electrodynamics is the fact that accelerated charges emit radiation. Here, we will derive the formula describing this result, “Larmor’s formula”, using a classical treatment due to J.J. Thomson and revived by Malcolm Longair. The ab initio derivation using Maxwell’s equations gives the same result.

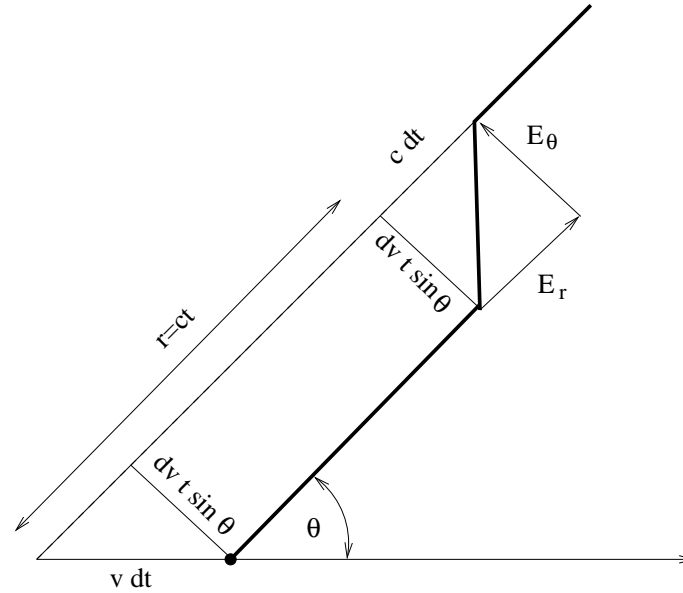
We start by taking a stationary charge at rest at time $t = 0$. The field lines from that charge have the simple configuration shown on the left side of the figure below.

We now accelerate the charge by a velocity difference Δv for a time interval Δt and look at the field line configuration at a time t after this. T



Outside of a sphere of radius $r \sim ct$ the field lines still point towards the original position of the charge, because the information about the acceleration has not yet moved farther out than r . Inside of this sphere the field lines point towards the location the charge had after the acceleration. Properly spoken, the figure is not correct as the charge continued to drift with its new velocity since the end of the acceleration. The field lines in the figure should reflect this, but given that that information has not yet propagated out to r either, we can ignore it, and all of the computations we do below will only need information local to r . Since the electric field lines have to be connected, there is a small region of width $c\Delta t$ in which the electric field has a non-radial component. An observer in this region will measure a temporal change in the E -field strength. By definition, a time dependent E -field is electromagnetic radiation and therefore this *Gedankenexperiment* has shown us that accelerating a charge will always lead to electromagnetic radiation!

To obtain the power radiated by the accelerated charge we need to know the strength of the electric field in the region of the electromagnetic pulse. This can be done by considering the following sketch of one E -field line:



where, for simplicity $dt = \Delta t$ and $dv = \Delta v$.

From simple geometry, one finds for the ratio of the electric fields in the pulse region:

$$\frac{E_{\theta}}{E_r} = \frac{\Delta v t \sin \theta}{c \Delta t} \quad (5.1)$$

E_r follows from Coulomb's law:

$$E_r = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{r^2} \quad (5.2)$$

and because we're observing at time t :

$$r = ct \quad (5.3)$$

Inserting E_r and r into Eq. (5.1) gives

$$E_{\theta} = E_r \frac{\Delta v t \sin \theta}{\Delta t} \frac{1}{c} = \frac{q}{4\pi\epsilon_0} \frac{1}{r} \frac{1}{ct} \frac{\Delta v t \sin \theta}{\Delta t} \frac{1}{c} \quad (5.4)$$

In the limit $\Delta t \rightarrow 0$, we can identify $\Delta v / \Delta t$ with the acceleration \dot{v} (where the dot denotes differentiation with respect to time), and therefore we finally obtain

$$E_{\theta} = \frac{q}{4\pi\epsilon_0} \frac{1}{rc^2} \dot{v} \sin \theta \quad (5.5)$$

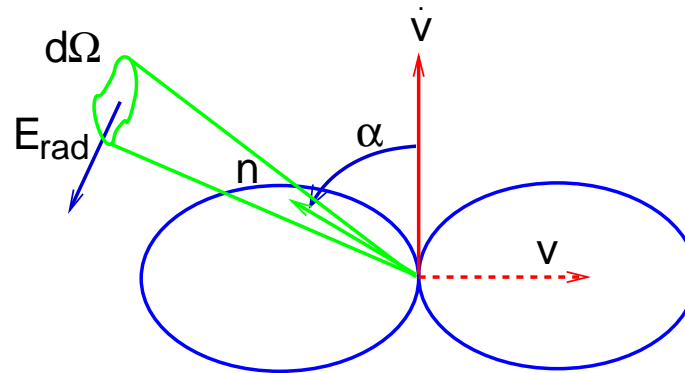
So, during the pulse the E -field in the θ -direction changes from 0 to E_θ and back to 0. This is a pulse of electromagnetic radiation. The energy flow per unit area and per second, S , is given from Poynting's law as

$$S = \epsilon_0 c E^2 \quad (5.6)$$

and therefore

$$S = \epsilon_0 c \frac{q^2}{16\pi^2 \epsilon_0^2} \frac{1}{c^4 r^2} \dot{v}^2 = \frac{1}{16\pi^2 c^3 \epsilon_0 r^2} q^2 \dot{v}^2 \sin^2 \theta \quad (5.7)$$

This means that the energy loss has *dipolar form*, as shown in the following figure. Often, this equation is called *Larmor's formula* (although we will encounter another Larmor's formula below).



Note that the energy loss is symmetric, this means that the radiating particle is only losing energy, but not momentum.

To obtain the total energy lost by the particle, we need to integrate the energy lost over all directions

$$\frac{dE}{dt} = \int_{4\pi \text{sr}} \frac{1}{16\pi^2 c^3 \epsilon_0^2 r^2} q^2 \dot{v}^2 \sin^2 \theta r^2 d\Omega \quad (5.8)$$

where $d\Omega = \sin\theta d\theta d\varphi$ is the surface element of a sphere, θ and φ are the usual spherical coordinates, with θ going from 0 to π and φ from 0 to 2π . The total surface area of a sphere is 4π steradians (abbreviated "sr"). Performing the integral, lumping all constants into a helper variable A gives

$$= A \int_{2\pi\text{sr}} \sin^2\theta d\Omega \quad (5.9)$$

$$= A \int_0^\pi \sin^2\theta 2\pi \sin\theta d\theta \quad (5.10)$$

$$= 2\pi A \int_0^\pi \sin^3\theta d\theta \quad (5.11)$$

$$= \frac{8\pi}{3} A \quad (5.12)$$

And therefore we obtain *Larmor's formula* for the energy loss of an accelerated charge:

$$\frac{dE}{dt} = \frac{8\pi}{3} \frac{1}{16\pi^2 c^3 \epsilon_0} q^2 \dot{v}^2 = \frac{q^2 \dot{v}^2}{6\pi c^3 \epsilon_0} \quad (5.13)$$