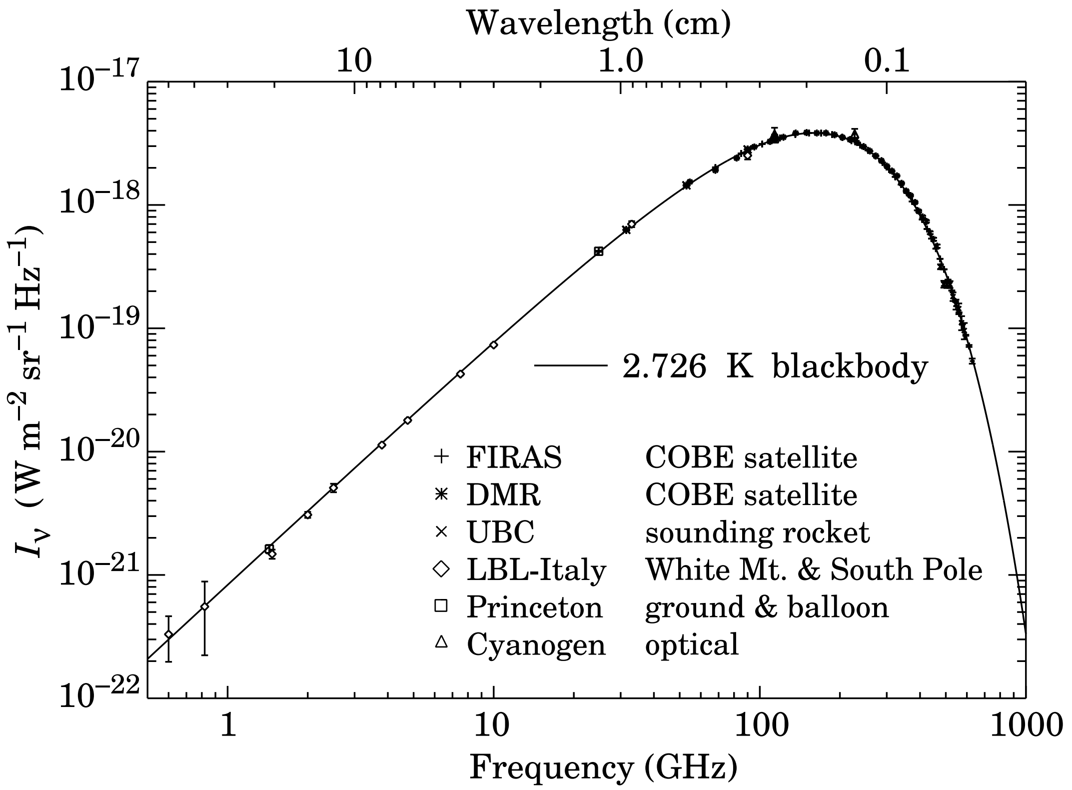
The Hot Big Bang

CMBR



(Smoot, 1997, Fig. 1)

Penzias & Wilson (1965): "Measurement of Excess Antenna Temperature at 4080 Mc/s"

$\Longrightarrow$ Cosmic Microwave Background Radiation (CMBR):

> The CMBR spectrum is fully consistent with a pure Planckian with temperature $T_{\text{CMBR}} = 2.728 \pm 0.004\,\text{K}$.

Now recognized as relict of hot big bang.

CMBR

Assumption: Early universe was hot and dense
$\Longrightarrow$ Equilibrium between matter and radiation.
Generation of radiation, e.g., in pair equilibrium,

$$\gamma + \gamma \longleftrightarrow \mathrm{e}^{-} + \mathrm{e}^{+} \tag{6.1}$$

Equilibrium with electrons, e.g., via Compton scattering:

$$\mathrm{e}^{-} + \gamma \longrightarrow \mathrm{e}^{-} + \gamma \tag{6.2}$$

where the electrons linked to protons via Coulomb interaction.
Once density low and temperature below photoionization for Hydrogen,

$$\mathrm{H} + \gamma \longleftrightarrow \mathrm{p} + \mathrm{e}^{-} \tag{6.3}$$

Decoupling of radiation and matter $\Longrightarrow$ Adiabatic cooling of photon field.
Proof for these assumptions, and lots of gory details: this and the next few lectures!

CMBR

Reminder: Planck formula for energy density of photons:

$$B_\lambda = \frac{\mathrm{d}u}{\mathrm{d}\lambda} = \frac{8\pi hc}{\lambda^5}\frac{1}{\exp(hc/k_\mathrm{B}T\lambda) - 1} \tag{6.4}$$

(units: $\mathrm{erg\,cm^{-3}\,\AA^{-1}}$), where

$$k_\mathrm{B} = 1.38 \times 10^{-16}\,\mathrm{erg\,K^{-1}} \quad \text{(Boltzmann)} \tag{6.5}$$

$$h = 6.625 \times 10^{-27}\,\mathrm{erg\,s} \quad \text{(Planck)} \tag{6.6}$$

For $\lambda \gg hc/k_\mathrm{B}T$: Rayleigh-Jeans formula:

$$B_\lambda \sim \frac{8\pi k_\mathrm{B}T}{\lambda^4} \tag{6.7}$$

(classical case, diverges for $\lambda \longrightarrow 0$, “Jeans catastrophe”).

Maximum emission given by Wien’s displacement law:

$$\lambda_\mathrm{max} = 0.201\frac{hc}{k_\mathrm{B}T} \tag{6.8}$$

Total energy density by integration:

$$u = \int_0^\infty B_\lambda\,\mathrm{d}\lambda = \frac{8\pi^5(kT)^4}{15h^3c^3} = \frac{4\sigma_\mathrm{SB}}{c}T^4 = a_\mathrm{rad}T^4 \tag{6.9}$$

where

$$\sigma_\mathrm{SB} = 5.670 \times 10^{-5}\,\mathrm{erg\,cm^{-3}\,K^{-4}} \quad \text{Stefan-Boltzmann} \tag{6.10}$$

$$a_\mathrm{rad} = 7.566 \times 10^{-15}\,\mathrm{erg\,cm^{-2}\,K^{-4}\,s^{-1}} \quad \text{rad. dens. const.} \tag{6.11}$$

CMBR

Since the energy of a photon is $E_\gamma = h\nu = hc/\lambda$, the number density of photons is

$$n = \int_0^\infty \frac{B_\lambda \, \mathrm{d}\lambda}{hc/\lambda} = 20.28\, T^3 \,\text{photons}\,\text{cm}^{-3} \tag{6.12}$$

Thus, for the CMBR:

$$n_{\text{CMBR}} = 400\,\text{photons}\,\text{cm}^{-3} \tag{6.13}$$

Compare that to baryons: $\Longrightarrow$ critical density:

$$\rho_{\text{c}} = \frac{3H^2}{8\pi G} = 1.88 \times 10^{-29} h^2 \text{ g cm}^{-3}$$
$$= 1.13 \times 10^{-5}\, h^2 \,\text{protons}\,\text{cm}^{-3} \tag{4.62}$$

since $m_{\text{p}} = 1.67 \times 10^{-24}$ g.
Therefore photons dominate the particle number:

$$\frac{n_{\text{CMBR}}}{n_{\text{baryons}}} = \frac{3.54 \times 10^7}{\Omega h^2} \tag{6.14}$$

But, baryons dominate the energy density:

$$\frac{u_{\text{CMBR}}}{u_{\text{baryons}}} = \frac{a_{\text{rad}} T^4}{\Omega \rho_{\text{c}} c^2} = \frac{4.20 \times 10^{-13}}{1.69 \times 10^{-8} \Omega h^2} = \frac{1}{40260\Omega h^2} \tag{6.15}$$

That's why we talk about the matter dominated universe.

CMBR

Remember the scaling laws for the (energy) density of matter and radiation:

$$\rho_{\text{m}} \propto R^{-3} \quad \text{and} \quad \rho_{\text{r}} \propto R^{-4} \qquad (4.67, 4.68)$$

Therefore,

$$\frac{\rho_{\text{r}}}{\rho_{\text{m}}} \propto \frac{1}{R} \qquad (6.16)$$

$\Longrightarrow$ Photons dominate for large z, i.e., early in the universe!

Since $1 + z = R_0/R$ (Eq. 4.43), matter-radiation equality was at

$$1 + z_{\text{eq}} = 40260\,\Omega h^2 \qquad (6.17)$$

(for $h = 0.75$, $1 + z_{\text{eq}} = 22650$)

The above definition of z_{eq} is not entirely correct: neutrino background, which increases the background energy density, is ignored ($u_\nu \sim 68\% u_\gamma$, see later).

Formally, matter-radiation equality defined from $n_{\text{baryons}} = n_{\text{rel. particles}}$, $\Longrightarrow$

$$1 + z_{\text{eq}} = 23900\,\Omega h^2 \qquad (6.18)$$

(for $h = 0.75$, $1 + z_{\text{eq}} = 13440$).

CMBR

What happened to the temperature of the CMBR? Compare CMBR spectrum today with earlier times.
Differential Energy density:

$$\mathrm{d}u = B_\lambda \mathrm{d}\lambda \tag{6.19}$$

Cosmological redshift:

$$\frac{\lambda'}{\lambda} = \frac{R'}{R} = \frac{1}{1+z} = a \tag{4.50}$$

where $R(\text{today}) = 1$.
Taking the expansion into account:

$$\begin{aligned}
\mathrm{d}u' = \frac{\mathrm{d}u}{a^4} &= \frac{8\pi hc}{a^4\lambda^5} \frac{\mathrm{d}\lambda}{\exp(hc/kT\lambda) - 1} && (6.20)\\
&= \frac{8\pi hc}{a^5\lambda^5} \frac{a\mathrm{d}\lambda}{\exp(hc/kT\lambda) - 1} && (6.21)\\
&= \frac{8\pi hc}{\lambda'^5} \frac{\mathrm{d}\lambda'}{\exp(hca/kT\lambda') - 1} && (6.22)\\
&= B_{\lambda'}(T/a) && (6.23)
\end{aligned}$$

Therefore, the Planckian remains a Planckian, and the temperature of the CMBR scales as

$$T(z) = (1+z)T_0 \tag{6.24}$$

The early universe was hot $\Longrightarrow$ Hot Big Bang Model!

Overview

$a(t)$	t since BB	T[K] [K]	ρ_{matter} [g cm^{-3}]	Major Events
	10^{-42}	10^{30}		Planck era, "begin of physics"
	$10^{-40\ldots-30}$	10^{25}		Inflation?
10^{-13}	$\sim 10^{-5}$ s	$\sim 10^{13}$	$\sim 10^{9}$	generation of p-p^{-}, and baryon anti-baryon pairs from radiation background
3×10^{-9}	1 min	10^{10}	0.03	generation of e^{+}-e^{-} pairs out of radiation background
10^{-9}	10 min	3×10^{9}	10^{-3}	nucleosynthesis
$10^{-4} \ldots 10^{-3}$	$10^{6\ldots7}$ yr	$10^{3\ldots4}$	$10^{-21\ldots-18}$	End of radiation dominated epoch
7×10^{-4}	10^{7} yr	4000	10^{-20}	Hydrogen recombines, decoupling of matter and radiation
1	15×10^{9} yr	3	10^{-30}	now

Thermodynamics, I

Density in early universe is very high.

physical processes (e.g., photon-photon pair creation, electron-positron annihilation etc.) all have reaction rates

$$\Gamma \propto n\sigma v \tag{6.25}$$

where
n: number density (cm^{-3})
σ: interaction cross-section (cm^{2})
v: velocity (cm s^{-1})
thermodynamic equilibrium reached if reaction rate much faster than "changes" in the system,

$$\Gamma \gg H \tag{6.26}$$

If thermodynamic equilibrium holds, then can assume evolution of universe as sequence of states of local thermodynamic equilibrium, and use standard thermodynamics.

Before looking at real universe, first need to derive certain useful formulae from relativistic thermodynamics.

Thermodynamics, II

For ideal gases, thermodynamics shows that number density $f(\mathbf{p})\,\mathrm{d}p$ of particles with momentum in $[p, p+\mathrm{d}p]$ is given by

$$f(\mathbf{p}) = \frac{1}{\exp\left((E-\mu)/k_{\mathrm{B}}T\right) + a} \tag{6.27}$$

where

$$a = \begin{cases} +1 & : \text{Fermions (spin=1/2, 3/2,...)} \\ -1 & : \text{Bosons (spin=1, 2,...)} \\ 0 & : \text{Maxwell-Boltzmann} \end{cases}$$

and where the energy needs to take the rest-mass into account:

$$E^2 = |\mathbf{p}|^2 c^2 + m^2 c^4 \tag{6.28}$$

μ is called the "chemical potential". It is preserved in chemical equilibrium:

$$i + j \leftrightarrow k + l \quad \Longrightarrow \quad \mu_i + \mu_j = \mu_k + \mu_l \tag{6.29}$$

photons: multi-photon processes exist $\Longrightarrow \mu_\gamma = 0$.
particles in thermal equilibrium: $\mu = 0$ as well because of the first law of thermodynamics,

$$\mathrm{d}E = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N \tag{6.30}$$

and in equilibrium system stationary wrt changes in particle number N.

Thermodynamics, III

In addition to number density: different particles have internal degrees of freedom, abbreviated with g.

Examples:

photons: two polarization states $\Longrightarrow g = 2$

neutrinos: one polarization state $\Longrightarrow g = 1$

electrons, positrons: spin=$1/2 \Longrightarrow g = 2$

Knowing g and $f(p)$, it is possible to compute interesting quantities:

particle number density:

$$n = \frac{g}{(2\pi\hbar)^3}\int f(\mathbf{p})\, \mathrm{d}^3p \tag{6.31}$$

energy density:

$$u = \rho c^2 = \frac{g}{(2\pi\hbar)^3}\int E(\mathbf{p})\, f(\mathbf{p})\, \mathrm{d}^3p \tag{6.32}$$

pressure: from kinetic theory we know

$$P = n\langle pv\rangle/3 = n\langle p^2c^2/E\rangle/3 \tag{6.33}$$

such that

$$P = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2c^2}{3E}\, f(\mathbf{p})\, \mathrm{d}^3p \tag{6.34}$$

Thermodynamics, IV

Generally, we are interested in knowing n, u, and P in two limiting cases:

1. the ultra-relativistic limit, where $k_{\mathrm{B}}T \gg mc^2$, i.e., kinetic energy dominates the rest-mass
2. the non-relativistic limit, where $k_{\mathrm{B}}T \ll mc^2$

Transitions between these limits (i.e., what happens during "cooling") are usually much more complicated $\Longrightarrow$ ignore...

To derive the number density, the energy density, and the equation of state, note that Eq. (6.28) shows

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{6.28}$$

such that

$$p = \sqrt{E^2 - m^2c^4}/c \tag{6.35}$$

Therefore

$$\frac{\mathrm{d}E}{\mathrm{d}p} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} \tag{6.36}$$

from which it follows that

$$E\,\mathrm{d}E = pc^2\,\mathrm{d}p \tag{6.37}$$

Thus the following holds

$$\iiint_{-\infty}^{+\infty} \mathrm{d}^3p = \int_0^\infty 4\pi p^2\,\mathrm{d}p = \int_{mc^2}^\infty \frac{4\pi}{c^3}\left(E^2 - m^2c^4\right)^{1/2} E\,\mathrm{d}E \tag{6.38}$$

Going to a system of units where

$$c = k_\mathrm{B} = \hbar = 1 \tag{6.39}$$

to save me some typing, substitute these equations into Eqs. (6.31)–(6.34) to find

$$n = \frac{g}{2\pi^2}\int_m^\infty \frac{\left(E^2 - m^2\right)^{1/2} E\,\mathrm{d}E}{\exp\left((E-\mu)/T\right) \pm 1} \tag{6.40}$$

$$\rho = \frac{g}{2\pi^2}\int_m^\infty \frac{\left(E^2 - m^2\right)^{1/2} E^2\,\mathrm{d}E}{\exp\left((E-\mu)/T\right) \pm 1} \tag{6.41}$$

$$P = \frac{g}{6\pi^2}\int_m^\infty \frac{\left(E^2 - m^2\right)^{3/2}\,\mathrm{d}E}{\exp\left((E-\mu)/T\right) \pm 1} \tag{6.42}$$

which can in some limiting cases be expressed in a closed form (Kolb & Turner, 1990, eq. 3.52 ff.) (see following viewgraphs).

Thermodynamics, V

In the ultra-relativistic limit, $k_{\rm B}T \gg mc^2$, and assuming $\mu = 0$,

$$n = \begin{cases} \frac{\zeta(3)}{\pi^2}\, g\, \left(\frac{k_{\rm B}T}{\hbar c}\right)^3 & \text{Bosons} \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2}\, g\, \left(\frac{k_{\rm B}T}{\hbar c}\right)^3 & \text{Fermions} \end{cases} \tag{6.43}$$

$$u = \begin{cases} \frac{\pi^2}{30}\, g\, k_{\rm B}T\, \left(\frac{k_{\rm B}T}{\hbar c}\right)^3 & \text{Bosons} \\ \frac{7}{8}\frac{\pi^2}{30}\, g\, k_{\rm B}T\, \left(\frac{k_{\rm B}T}{\hbar c}\right)^3 & \text{Fermions} \end{cases} \tag{6.44}$$

$$P = \rho c^2/3 = u/3 \tag{6.45}$$

where $\zeta(3) = 1.202\ldots$, and $\zeta(s)$ is Riemann's zeta-function (see handout, Eq. 6.53).

Eq. (6.45) is a simple result of the fact that in the relativistic limit, $E \sim pc$. Inserting this and $v = c$ into Eq. (6.33) gives the desired result.

As expected, T^4 proportionality well known from Stefan Boltzmann law!

Obtaining the previous formulae is an exercise in special functions. For example, the $T \gg m$, $T \gg \mu$ case for ρ for Bosons (Eq. 6.44) is obtained as follows (setting $c = k_\mathrm{B} = \hbar = 1$):

$$\rho_\text{Boson} = \frac{g}{2\pi^2} \int_m^\infty \frac{\left(E^2 - m^2\right)^{1/2} E^2 \, \mathrm{d}E}{\exp\left((E-\mu)/T\right) \pm 1} \tag{6.46}$$

because of $T \gg \mu$

$$\approx \frac{g}{2\pi^2} \int_m^\infty \frac{\left(E^2 - m^2\right)^{1/2} E^2 \, \mathrm{d}E}{\exp(E/T) \pm 1} \tag{6.47}$$

for *Bosons*, choose -1, and substitute $x = E/T$:

$$= \frac{g}{2\pi^2} \int_{m/T}^\infty \frac{\left(x^2T^2 - m^2\right)^{1/2} x^2 T^3 \, \mathrm{d}x}{\exp(x) - 1} \tag{6.48}$$

Since $T \gg m$,

$$\approx \frac{g}{2\pi^2} \int_0^\infty \frac{x^3 T^4 \, \mathrm{d}x}{\exp(x) - 1} \tag{6.49}$$

$$= \frac{gT^4}{2\pi^2} \int_0^\infty \frac{x^3 \, \mathrm{d}x}{\exp(x) - 1} \tag{6.50}$$

$$= \frac{gT^4}{2\pi^2} \cdot 6\zeta(4) \tag{6.51}$$

$$= \frac{\pi^2}{30} gT^4 \tag{6.52}$$

where $\zeta(s)$ is *Riemann's zeta-function*, which is defined by (Abramowitz & Stegun, 1964)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{\exp(x) - 1} \, \mathrm{d}x \quad \text{for } \mathscr{R}e\, s > 1 \tag{6.53}$$

where $\Gamma(x)$ is the Gamma-function. Note that $\zeta(4) = \pi^4/90$.

For *Fermions*, everything is the same except for that we now have to choose the $+$ sign. The equivalent of Eq. (6.50) is then

$$\rho_\text{Fermi} = \frac{gT^4}{2\pi^2} \int_0^\infty \frac{x^3 \, \mathrm{d}x}{\exp(x) + 1} \tag{6.54}$$

Now we can make use of formula 3.411.3 of Gradstein & Ryshik (1981),

$$\int_0^\infty \frac{x^{\nu-1} \, \mathrm{d}x}{\exp(\mu x) + 1} = \frac{1}{\mu^\nu}(1 - 2^{1-\nu})\Gamma(\nu)\zeta(\nu) \quad \text{for } \mathscr{R}e\, \mu, \nu > 1 \tag{6.55}$$

to see where the additional factor of $7/8$ in Eq. (6.44) comes from.

Thermodynamics, VI

In the non-relativistic limit: $k_\mathrm{B}T \ll mc^2$

$\Longrightarrow$ can ignore the ± 1 term in the denominator

$\Longrightarrow$ Same formulae for Bosons and Fermions!

$$n = \frac{2g}{(2\pi\hbar)^3}(2\pi m k_\mathrm{B} T)^{3/2} \mathrm{e}^{-mc^2/k_\mathrm{B}T} \qquad (6.56)$$

$$u = nmc^2 \qquad (6.57)$$

$$P = nk_\mathrm{B}T \qquad (6.58)$$

Therefore:

- density dominated by rest-mass ($\rho = u/c^2 = mn$)
- $P \ll \rho c^2/3$, i.e., *much* smaller than for relativistic particles.
- Particle pressure only important if particles are relativistic.

Obviously, relativistic particles with $m = 0$ (or very close to 0) will never get nonrelativistic. Still, they can “decouple” from the rest of the universe when the interaction rates go to 0.

Equation of State

Pressure of ultra-relativistic particles $\gg$ Pressure of nonrelativistic particles $\Longrightarrow$ Nonrelativistic particles unimportant for equation of state.

For relativistic particles:

$$u_{\text{boson}} = \frac{\pi^2}{30} g\, k_{\text{B}}T \left(\frac{k_{\text{B}}T}{\hbar c}\right)^3 \qquad (6.44)$$

$$u_{\text{fermion}} = \frac{7}{8} u_{\text{boson}} \qquad (6.44)$$

$\Longrightarrow$ Total energy density for mixture of particles:

$$u = g_* \cdot \frac{\pi^2}{30} k_{\text{B}}T \left(\frac{k_{\text{B}}T}{\hbar c}\right)^3 \qquad (6.59)$$

where the effective degeneracy factor

$$g_* = \sum_{\text{bosons}} g_{\text{B}} \left(\frac{T_{\text{B}}}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_{\text{F}} \left(\frac{T_{\text{F}}}{T}\right)^4 \qquad (6.60)$$

g_* counts total number of internal degrees of freedom of *all* relativistic bosonic and fermionic species, i.e., all relativistic particles which are in thermodynamic equilibrium

Pressure obtained from Eq. (6.59) via $P = u/3$.

Early Expansion, I

Knowing the equation of state (EOS), we can now use Friedmann equations to determine the early evolution of the universe.

Friedmann:

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 \tag{4.59}$$

or, dividing by R^2

$$\frac{\dot{R}^2}{R^2} = H(t)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} \tag{4.60}$$

But: Early universe dominated by relativistic particles

$\Longrightarrow \rho \propto R^{-4}$

$\Longrightarrow$ Density-term dominates

$\Longrightarrow$ can set $k = 0$.

Early universe is asymptotically flat!

This will prove to be one of the most crucial problems of modern cosmology. . .

Early Expansion, II

To obtain evolution, insert EOS (Eq. 6.59) into Eq. (4.60):

$$H(t)^2 = \frac{8\pi G}{3} g_* \frac{\pi^2}{30} \frac{(k_\mathrm{B}T)^4}{(\hbar c)^3} \tag{6.61}$$

$$= \frac{4\pi^3 G}{45(\hbar c)^3} g_* (k_\mathrm{B}T)^4 \tag{6.62}$$

such that

$$H(t) = \left(\frac{4\pi^3 G}{45(\hbar c)^3}\right)^{1/2} g_*^{1/2} (k_\mathrm{B}T)^2 \tag{6.63}$$

On the other hand, since $\rho \propto R^{-4}$ (relativistic background),

$$\rho = \rho_0 \left(\frac{R_0}{R}\right)^4 \tag{6.64}$$

Friedmann:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sqrt{\frac{8\pi G\rho_0}{3}\frac{R_0^2}{R}} \tag{6.65}$$

Introducing the dimensionless scale factor, $a = R/R_0$ (Eq. 4.30),

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{8\pi G\rho_0}{3}\frac{1}{a}} =: \xi a^{-1} \tag{6.66}$$

Early Expansion, III

Separation of variables gives

$$\int_0^{a(t)} a \, \mathrm{d}a = \int_0^t \xi \, \mathrm{d}t \tag{6.67}$$

such that finally

$$a(t) = \xi^{1/2} \cdot t^{1/2} \tag{6.68}$$

Therefore, the Hubble constant is

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{2t} \tag{6.69}$$

Equating Eqs. (6.63) and (6.69) gives the time-temperature relationship:

$$t = \left(\frac{45(\hbar c)^3}{16\pi^3 G}\right)^{1/2} \frac{1}{g_*^{1/2}} \frac{1}{(k_\mathrm{B}T)^2} \tag{6.70}$$

Inserting all constants and converting to more useful units gives

$$\boxed{t = \frac{2.4\,\mathrm{sec}}{g_*^{1/2}} \cdot \left(\frac{k_\mathrm{B}T}{1\,\mathrm{MeV}}\right)^{-2}} \tag{6.71}$$

. . . one of the most useful equations for the early universe.

Elementary Particles, I

Precise behavior of universe depends on g_*
$\Longrightarrow$ Strong dependency on elementary particle physics.

Generally, particles present when energy in other particles allows generation of particle–antiparticle pairs, i.e., when $k_\mathrm{B}T \gtrsim mc^2$ (threshold temperature)

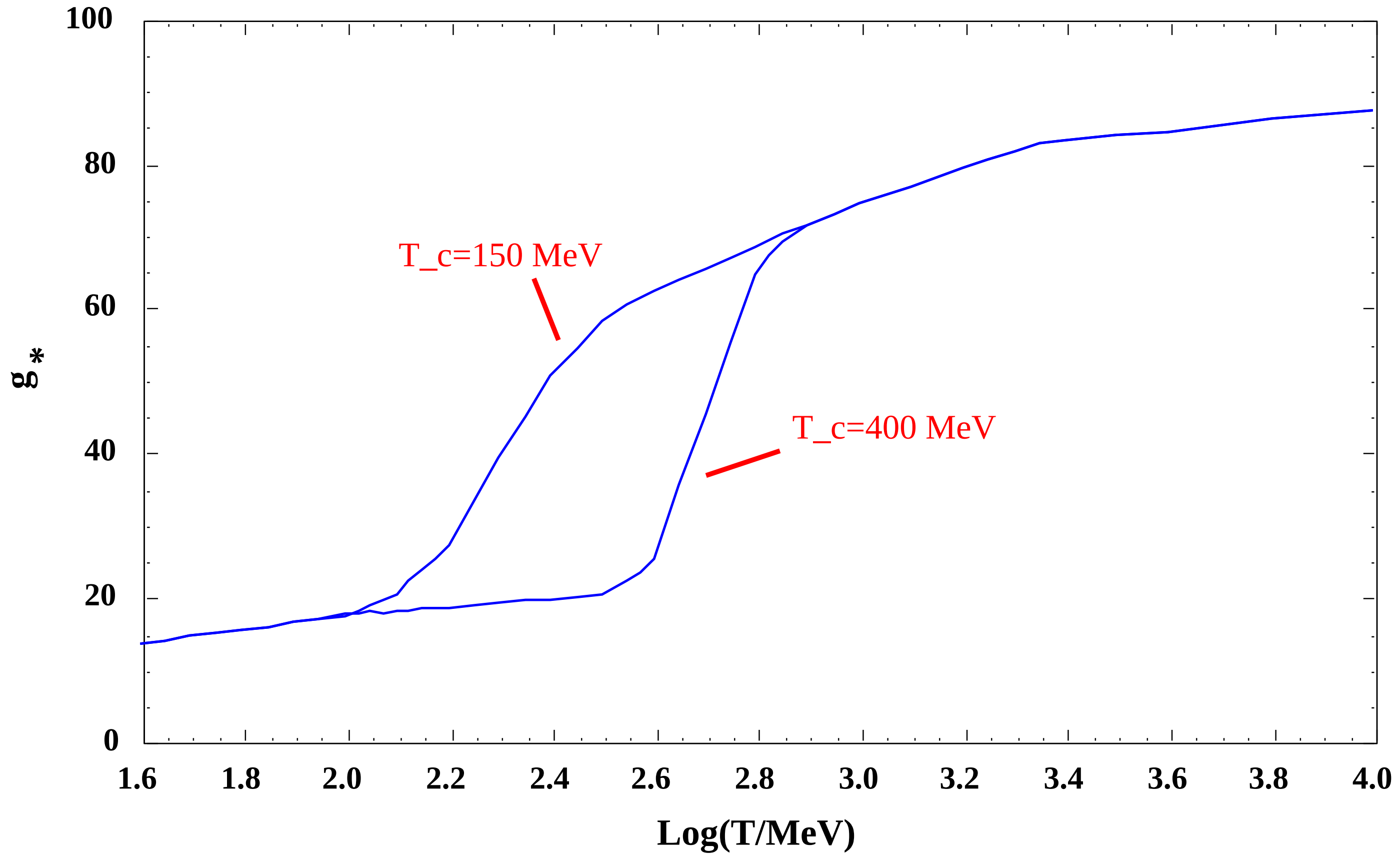
Current particle physics provides following picture (Olive, 1999, Tab. 1):

Temp.	New Particles	$4g_*(T)$
$k_\mathrm{B}T < m_\mathrm{e}c^2$	γ's and ν's	29
$m_\mathrm{e}c^2 < k_\mathrm{B}T < m_\mu$	$\mathrm{e}^\pm$	43
$m_\mu c^2 < k_\mathrm{B}T < m_\pi$	$\mu^\pm$	57
$m_\pi c^2 < k_\mathrm{B}T < k_\mathrm{B}T_\mathrm{c}$	π's	69
$k_\mathrm{B}T_\mathrm{c} < k_\mathrm{B}T < m_\mathrm{strange}c^2$	$-\pi$'s$+$u, $\bar{\mathrm{u}}$, d, $\bar{\mathrm{d}}$, gluons	205
$m_\mathrm{s}c^2 < k_\mathrm{B}T < m_\mathrm{charm}c^2$	s, $\bar{\mathrm{s}}$	247
$m_\mathrm{c}c^2 < k_\mathrm{B}T < m_\tau c^2$	c, $\bar{\mathrm{c}}$	289
$m_\tau c^2 < k_\mathrm{B}T < m_\mathrm{bottom}c^2$	$\tau^\pm$	303
$m_\mathrm{b}c^2 < k_\mathrm{B}T < m_\mathrm{W,Z}c^2$	b, $\bar{\mathrm{b}}$	345
$m_\mathrm{W,Z}c^2 < k_\mathrm{B}T < m_\mathrm{top}c^2$	$\mathrm{W}^\pm$, Z	381
$m_\mathrm{t}c^2 < k_\mathrm{B}T < m_\mathrm{Higgs}c^2$	t, $\bar{\mathrm{t}}$	423
$m_\mathrm{H}c^2 < k_\mathrm{B}T$	H^0	427

T_c: energy of confinement-deconfinement for transitions quarks $\Longrightarrow$ hadrons, somewhere between 150 MeV and 400 MeV.

Example: photons (2 polarization states, i.e., $g = 2$) and three species of neutrinos ($g = 1$, but with distinguishable anti-particles)
$\Longrightarrow g_* = 2 + (7/8) \cdot 2 \cdot 3 = 58/8 = 29/4$.

Elementary Particles, II



(Olive, 1999, Fig. 1)

Will now consider times when only Neutrinos and Electron/Positrons present (after baryogenesis, see next lecture for that).

Interlude

Previous (abstract) formulae allow to estimate quantities like

1. The existence and energy of primordial neutrinos,
2. The formation of neutrons,
3. The formation of heavier elements.

Detailed computations require solving nonlinear differential equations $\Longrightarrow$ difficult, only numerically possible.

Essentially, need to self-consistently solve Boltzmann equation in expanding universe for evolution of phase space density with time, using the correct QCD/QED reaction rates $\Longrightarrow$ too complicated (at least for me...).

Will use approximate analytical way here, which gives surprisingly exact answers.

Neutrinos, I

Neutrino equilibrium caused by weak interactions such as

$$\begin{aligned} \mathrm{e}^- + \mathrm{e}^+ &\longleftrightarrow \nu + \bar{\nu} \\ \mathrm{e}^- + \nu &\longleftrightarrow \mathrm{e}^- + \nu \end{aligned} \tag{6.72}$$

etc.

Reaction rate for these processes:

$$\Gamma = n \langle \sigma v \rangle \tag{6.73}$$

where the thermally averaged interaction cross-section is

$$\langle \sigma v \rangle \approx \left\langle \frac{\alpha^2 p}{m_\mathrm{W}^4} \cdot p \right\rangle \sim 10^{-2} \frac{(k_\mathrm{B} T)^2}{m_\mathrm{W}^4} \tag{6.74}$$

m_W: mass of W-boson (exchange particle of weak interaction),
$\alpha \approx 1/137$: fine structure constant.

But in the ultra-relativistic limit,

$$n \propto T^3 \tag{6.43}$$

such that

$$\Gamma_\mathrm{weak} \propto \frac{\alpha^2 T^5}{m_\mathrm{W}^4} \tag{6.75}$$

Neutrinos, II

Because of Eqs. (6.69) and (6.70), the temperature dependence of the Hubble constant is

$$H(T) = 1.66 g_*^{1/2} \cdot \frac{T^2}{m_\mathrm{P}} \tag{6.76}$$

where m_P is the Planck mass, $m_\mathrm{P}c^2 = 1.22 \times 10^{19}\,\mathrm{GeV}$ (see later, Eq. 6.130).

Neutrino equilibrium possible as long as $\Gamma_\mathrm{weak} > H$, i.e., (inserting exact numbers)

$$k_\mathrm{B}T_\mathrm{dec} \gtrsim \left(\frac{500\, c^6\, m_\mathrm{W}^4}{m_\mathrm{P}}\right)^{1/3} \sim 1\,\mathrm{MeV} \tag{6.77}$$

Neutrinos decouple $\sim 1\,\mathrm{s}$ after the big bang.

This follows from Eq. (6.71), remembering that for this phase, $g_* \sim 10$.

Since decoupling, primordial neutrinos just follow expansion of universe, virtually no interaction with “us” anymore.

Entropy, I

The entropy of particles is defined through

$$S = \frac{E + PV}{T} \tag{6.78}$$

Important for cosmology: relativistic limit.

Define entropy density,

$$s = \frac{S}{V} = \frac{E/V + P}{T} = \frac{u + P}{T} \approx \frac{4}{3}\frac{u}{T} \tag{6.79}$$

(last step for relativistic limit; Eq. 6.45)

Inserting Eq. (6.44) gives

$$s = \frac{7}{8}\frac{2\pi^2}{45} g k_\mathrm{B} \left(\frac{k_\mathrm{B}T}{\hbar c}\right)^3 = \frac{7}{6}\frac{2\pi^4}{45\,\zeta(3)} k_\mathrm{B}\, n \tag{6.80}$$

(violet: only for Fermions).

$\Longrightarrow$ In the relativistic limit

$$\frac{s}{k_\mathrm{B}} = \begin{cases} 3.602n & \text{Bosons} \\ 4.202n & \text{Fermions} \end{cases} \tag{6.81}$$

Important for later:

Since $s \propto n$ for backgrounds,
$\eta = n_\mathrm{CMBR}/n_\mathrm{baryons}$ is often called "entropy per baryon".

Entropy, II

For a mixture of backgrounds, Eq. (6.80) gives

$$\frac{s}{k_\mathrm{B}} = g_{*,S} \cdot \frac{2\pi^2}{45} \left(\frac{k_\mathrm{B}T}{\hbar c}\right)^3 \qquad (6.82)$$

where $g_{*,S}$ is the analogue to g_* (Eq. 6.60),

$$g_{*,S} = \sum_{\text{bosons}} g_\mathrm{B} \left(\frac{T_\mathrm{B}}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_\mathrm{F} \left(\frac{T_\mathrm{F}}{T}\right)^3 \qquad (6.83)$$

Note that if the species are not at the same temperature, $g_* \neq g_{*,S}$.

Entropy per mass today:

$$\frac{S}{M} = \frac{10^{16}}{\Omega h^2} \,\mathrm{erg\,K^{-1}\,g^{-1}} \qquad (6.84)$$

while the entropy gain of heating water at 300 K by 1 K is $\sim 1.4 \times 10^5\,\mathrm{erg\,K^{-1}\,g^{-1}}$.

$\Longrightarrow$ “Human attempts to obey 2nd law . . . are swamped by . . . microwave background” (Peacock, 1999, p. 277).

$\Longrightarrow$ $S =$ const. for universe to very good approximation.

$\Longrightarrow$ Universe expansion is adiabatic!

Reheating

After decoupling of neutrinos, neutrino distribution just gets redshifted (similar to CMBR, Eq. 6.24):

$$\frac{T_\nu}{T_{\text{dec}}} = \frac{R_{\text{dec}}}{R(t)} \quad \Longrightarrow \quad T_\nu \propto R^{-1} \tag{6.85}$$

On the other hand, the temperature of the universe is

$$T \propto g_{*,S}^{1/3}\, R^{-1} \tag{6.86}$$

This follows from $S/V \propto T^3$ (Eq. 6.82), $V \propto R^3$, and $S = \text{const.}$ (adiabatic expansion of the universe).

$\Longrightarrow$ as long as $g_{*,S} = \text{const.}$ we have $T_\nu = T$

$\Longrightarrow$ Immediately after decoupling, neutrino background appears as if it is still in equilibrium.

However: Temperature for neutrino decoupling $\sim 2m_{\text{e}}c^2$

But, for $kT_{\text{BB}} < 2\, m_{\text{e}}c^2$, pair creation,

$$\gamma + \gamma \longleftrightarrow \text{e}^- + \text{e}^+ \tag{6.87}$$

kinematically impossible

$\Longrightarrow$ Shortly after neutrino decoupling: $\text{e}^\pm$ annihilation

$\Longrightarrow$ $g_{*,S}$ changes!

$\Longrightarrow$ Would expect $T_{\text{CMBR}} \neq T_\nu$.

Reheating

Difference in $g_{*,S}$:

- before annihilation:
  e^-, e^+, $\gamma \Longrightarrow g_{*,S} = 2 + 2 \cdot 2 \cdot (7/8) = 11/2$.
- after annihilation:
  $\gamma \Longrightarrow g_{*,S} = 2$

But: total entropy for particles in equilibrium conserved ("expansion is adiabatic"):

$$g_{*,S}(T_{\text{before}}) \cdot T^3_{\text{before}} = g_{*,S}(T_{\text{after}}) \cdot T^3_{\text{after}} \qquad (6.88)$$

such that

$$T_{\text{after}} = \left(\frac{11}{4}\right)^{1/3} T_{\text{before}} \sim 1.4 \cdot T_{\text{before}} \qquad (6.89)$$

Since $T_{\text{after}} > T_{\text{before}}$: "reheating".

Note that in reality the annihilation is not instantaneous and T decreases (albeit less rapidly) during "reheating"...

$\Longrightarrow$ Since neutrino-background does not "see" annihilation

$\Longrightarrow$ just continues to cool

$\Longrightarrow$ current temperature of neutrinos is

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_{\text{CMBR}} \sim 1.95\,\text{K} \qquad (6.90)$$

History

After reheating: universe consists of p, n, γ (and e^- to preserve charge neutrality)
$\Longrightarrow$ Ingredients for Big Bang Nucleosynthesis (BBN).

Historical perspective:

Cross section to make Deuterium:

$$\langle\sigma v\rangle(\mathrm{p}+\mathrm{n}\longrightarrow \mathrm{D}+\gamma)\sim 5\times 10^{-20}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \tag{6.91}$$

Furthermore, need temperatures of $T_{\mathrm{BBN}}\sim 100\,\mathrm{keV}$, i.e., $t_{\mathrm{BBN}}\sim 200\,\mathrm{s}$ (Eq. 6.71).
This implies density

$$n\sim\frac{1}{\langle\sigma v\rangle\cdot t_{\mathrm{BBN}}}\sim 10^{17}\,\mathrm{cm}^{-3} \tag{6.92}$$

Today: Baryon density $n_{\mathrm{B}}\sim 10^{-7}\,\mathrm{cm}^{-3}$
Since $n\propto R^{-3}$, $\Longrightarrow$

$$T(\mathrm{today})=\left(\frac{n_{\mathrm{B}}}{n}\right)^{1/3}\cdot T_{\mathrm{BBN}}\sim 10\,\mathrm{K} \tag{6.93}$$

pretty close to the truth...

The above discussion was first used by Gamov and coworkers in 1948, and was the first prediction of the cosmic microwave background radiation!

Observations:

BBN required by observations, since no other production region for Deuterium known, and since He-abundance $\sim$ 25% by mass everywhere.

Proton/Neutron

Initial conditions: Set by Proton-Neutron-Ratio.

For $t \ll 1\,\mathrm{s}$, equilibrium via weak interactions:

$$\begin{aligned} \mathrm{n} &\longleftrightarrow p + \mathrm{e}^- + \bar{\nu}_\mathrm{e} \\ \nu_\mathrm{e} + \mathrm{n} &\longleftrightarrow p + \mathrm{e}^- \\ \mathrm{e}^+ + \mathrm{n} &\longleftrightarrow p + \bar{\nu}_\mathrm{e} \end{aligned} \tag{6.94}$$

Reactions fast as long as particles relativistic.

But, once $T \sim 1\,\mathrm{MeV}$, n, p non-relativistic
$\Longrightarrow$ Boltzmann statistics applies (or us Eq. (6.56)):

$$\frac{n_\mathrm{n}}{n_\mathrm{p}} = \mathrm{e}^{-\Delta mc^2/k_\mathrm{B}T} = \mathrm{e}^{-1.3\,\mathrm{MeV}/k_\mathrm{B}T} \tag{6.95}$$

$\Longrightarrow$ Suppression of n with respect to p because of larger mass ($m_\mathrm{n}c^2 = 939.57\,\mathrm{MeV}$, $m_\mathrm{p}c^2 = 938.27\,\mathrm{MeV}$)

Abundance freezes out when $\Gamma > H$, where reaction rate

$$\Gamma(\nu_\mathrm{e} + \mathrm{n} \leftrightarrow \mathrm{p} + \mathrm{e}^-) \sim 2.1 \left(\frac{T}{1\,\mathrm{MeV}}\right)^5 \mathrm{s}^{-1} \tag{6.96}$$

Neutron abundance freezes out at $k_\mathrm{B}T \sim 0.8\,\mathrm{MeV}$ ($t = 1.7\,\mathrm{s}$), such that $n_\mathrm{n}/n_\mathrm{p} = 0.2$

After that: Neutron decay ($\tau_\mathrm{n} = 886.7 \pm 1.2\,\mathrm{s}$).

$\Longrightarrow$ Nucleosynthesis has to be over before neutrons are gone!

Deuterium

The next step in nucleosynthesis is formation of deuterium (binding energy $E_\mathrm{B} = 2.225\,\mathrm{MeV}$, i.e., $1.7(m_\mathrm{n} - m_\mathrm{p})c^2$:

$$\mathrm{p} + \mathrm{n} \longleftrightarrow \mathrm{D} + \gamma \tag{6.97}$$

Note: Both reactions possible:
fusion and photodisintegration:

$$\Gamma_\mathrm{fusion} = n_\mathrm{B}\sigma v \tag{6.98}$$

$$\Gamma_\mathrm{photo} = n_\gamma \sigma v \mathrm{e}^{-E_\mathrm{B}/k_\mathrm{B}T} \tag{6.99}$$

At first: photodisintegration dominates ($\eta^{-1} = n_\gamma/n_\mathrm{B} \sim 10^{10}$).
Build up of D only possible once $\Gamma_\mathrm{fusion} > \Gamma_\mathrm{photo}$, i.e., when

$$\frac{n_\gamma}{n_\mathrm{B}} \mathrm{e}^{-E_\mathrm{B}/k_\mathrm{B}T} \sim 1 \tag{6.100}$$

Inserting numbers shows that

Deuterium production starts at $k_\mathrm{B}T \sim 100\,\mathrm{keV}$, $t \sim 100\,\mathrm{s}$.

Heavier Elements, I

Once deuterium present:

nucleosynthesis of lighter elements:

$$\begin{aligned} \mathrm{D}+\mathrm{D} &\longrightarrow \mathrm{T}+\mathrm{p} \\ \mathrm{D}+\mathrm{n} &\longrightarrow \mathrm{T}+\gamma \\ \mathrm{D}+\mathrm{p} &\longrightarrow {}^3\mathrm{He}+\gamma \\ \mathrm{D}+\mathrm{D} &\longrightarrow {}^3\mathrm{He}+\mathrm{n} \\ {}^3\mathrm{He}+\mathrm{n} &\longrightarrow \mathrm{T}+\mathrm{p} \end{aligned} \tag{6.101}$$

production of ^{4}He:

$$\begin{aligned} \mathrm{D}+\mathrm{D} &\longrightarrow {}^4\mathrm{He}+\gamma \\ \mathrm{D}+{}^3\mathrm{He} &\longrightarrow {}^4\mathrm{He}+\mathrm{p} \\ \mathrm{T}+\mathrm{D} &\longrightarrow {}^4\mathrm{He}+\mathrm{n} \\ {}^3\mathrm{He}+{}^3\mathrm{He} &\longrightarrow {}^4\mathrm{He}+2\mathrm{p} \\ \mathrm{T}+\mathrm{p} &\longrightarrow {}^4\mathrm{He}+\gamma \\ {}^3\mathrm{He}+\mathrm{n} &\longrightarrow {}^4\mathrm{He}+\gamma \end{aligned} \tag{6.102}$$

Element gap at $A = 5$ can be overcome to produce Lithium:

$$\begin{aligned} {}^3\mathrm{He}+{}^4\mathrm{He} &\longrightarrow {}^7\mathrm{Be}+\gamma \\ {}^7\mathrm{Be} &\longrightarrow {}^7\mathrm{Li}+\mathrm{e}^+ +\nu_\mathrm{e} \\ \mathrm{T}+{}^4\mathrm{He} &\longrightarrow {}^7\mathrm{Li}+\mathrm{e}^+ +\nu_\mathrm{e} \end{aligned} \tag{6.103}$$

Gap at $A = 8$ prohibits production of heavier isotopes.

Heavier Elements, II

Major product of BBN: ^{4}He.

Mass fraction of ^{4}He assuming all neutrons incorporated into ^{4}He

$\Longrightarrow$ number density of H=number of remaining protons, i.e., mass fraction

$$X = \frac{n_p - n_n}{n_p + n_n} \quad (6.104)$$

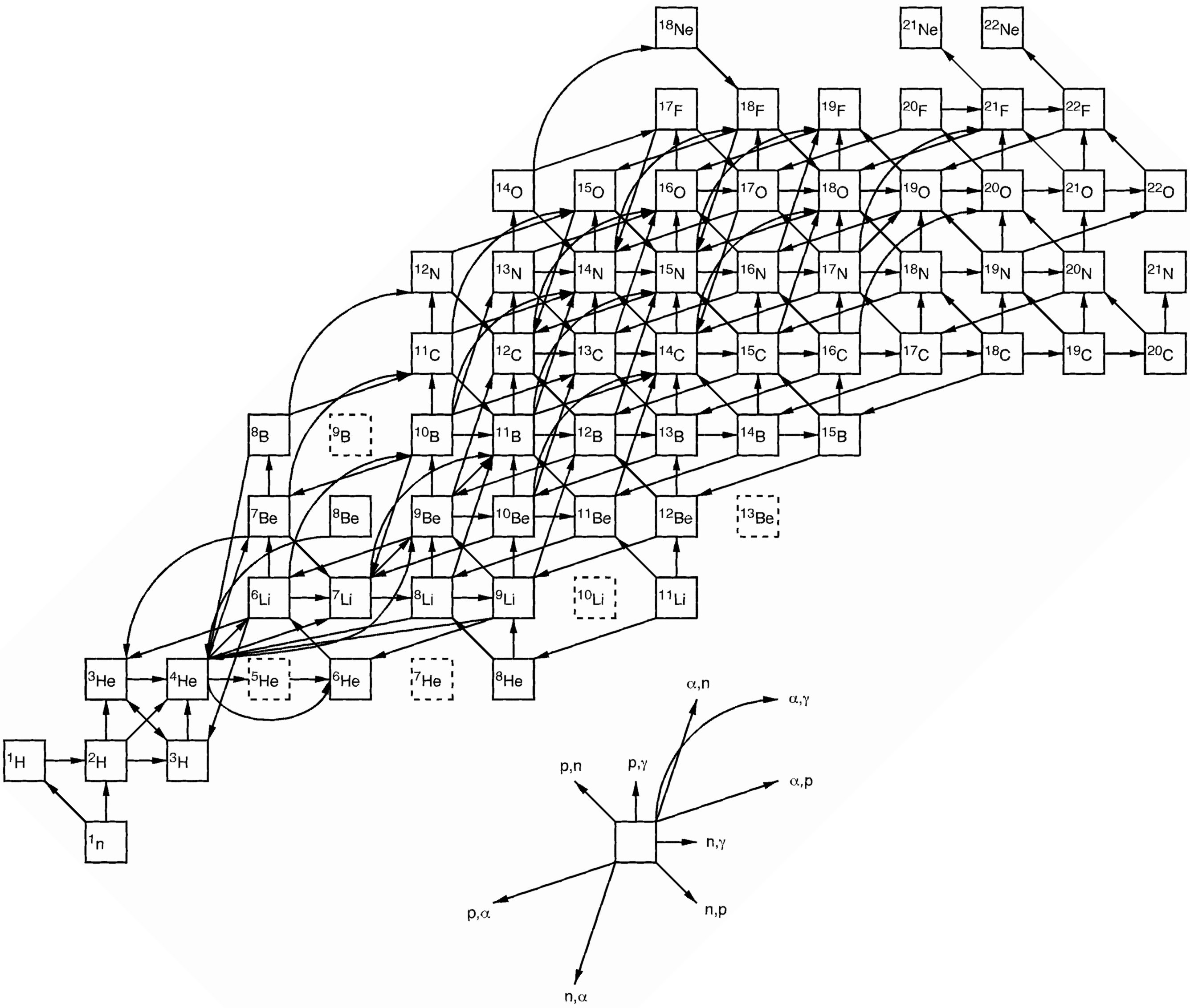
and

$$Y = 1 - \frac{n_p - n_n}{n_p + n_n} = 2\left(1 + \frac{n_p}{n_n}\right)^{-1} \quad (6.105)$$

At $k_B T = 0.8\,\text{MeV}$, because of neutron decay, $n_n/n_p = 1/7$, therefore

BBN predicts primordial He-abundance of $Y = 0.25$.

1. Generally, BBN function of entropy per baryon, η, i.e., of Ω_B:
   $$\Omega_B = 3.67 \times 10^7 \cdot \eta \quad (6.106)$$
   (since η, Ω determine expansion behavior) $\Longrightarrow$ Perform computations as function of η!
2. Since Y set by n_p/n_n $\Longrightarrow$ Relatively independent on η (except for extreme values).



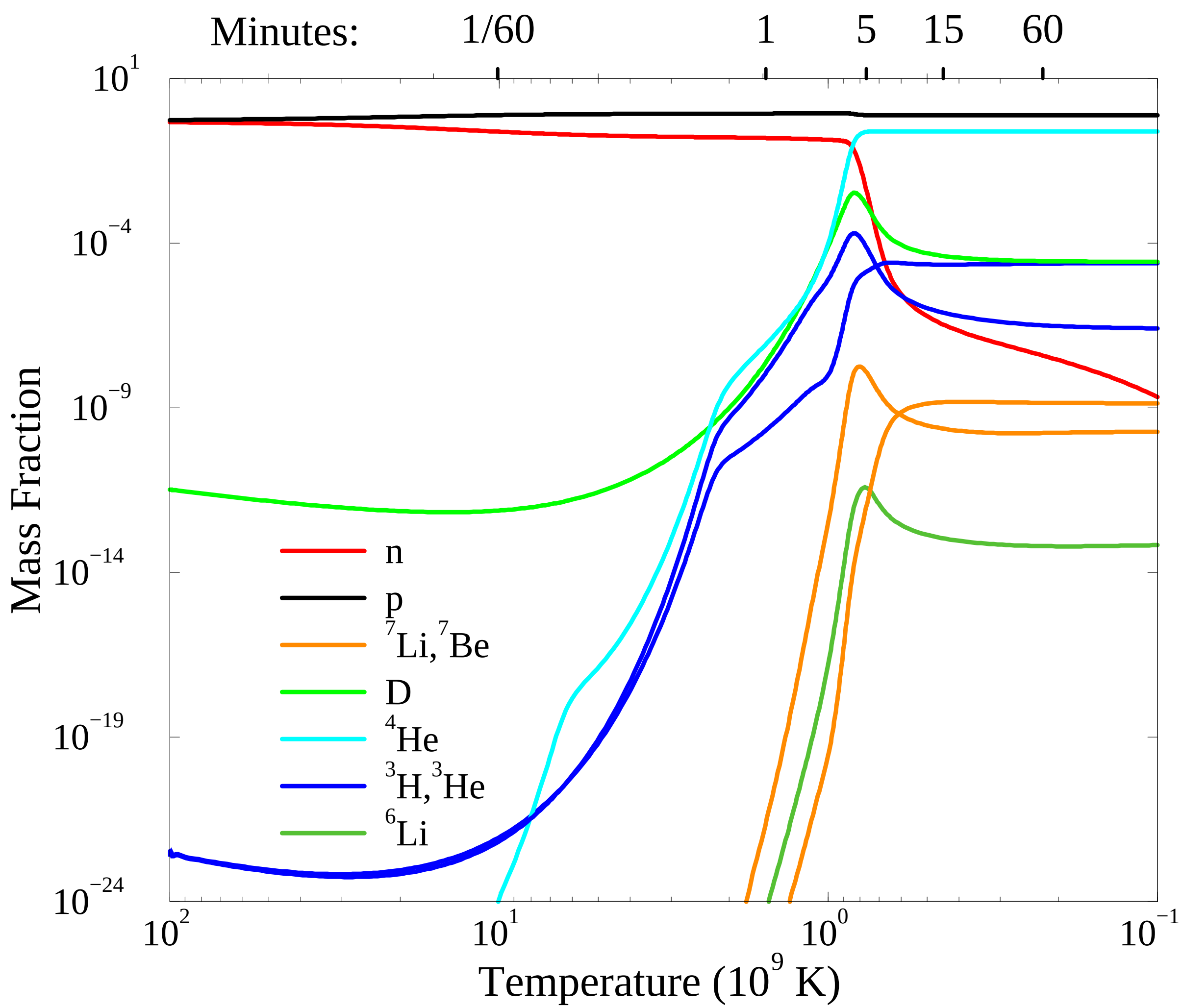
(Olive, 1999, Fig. 3)

Detailed Computations: Solution of rate-equations in expanding universe.

Recent computations: Thomas et al. (1993).

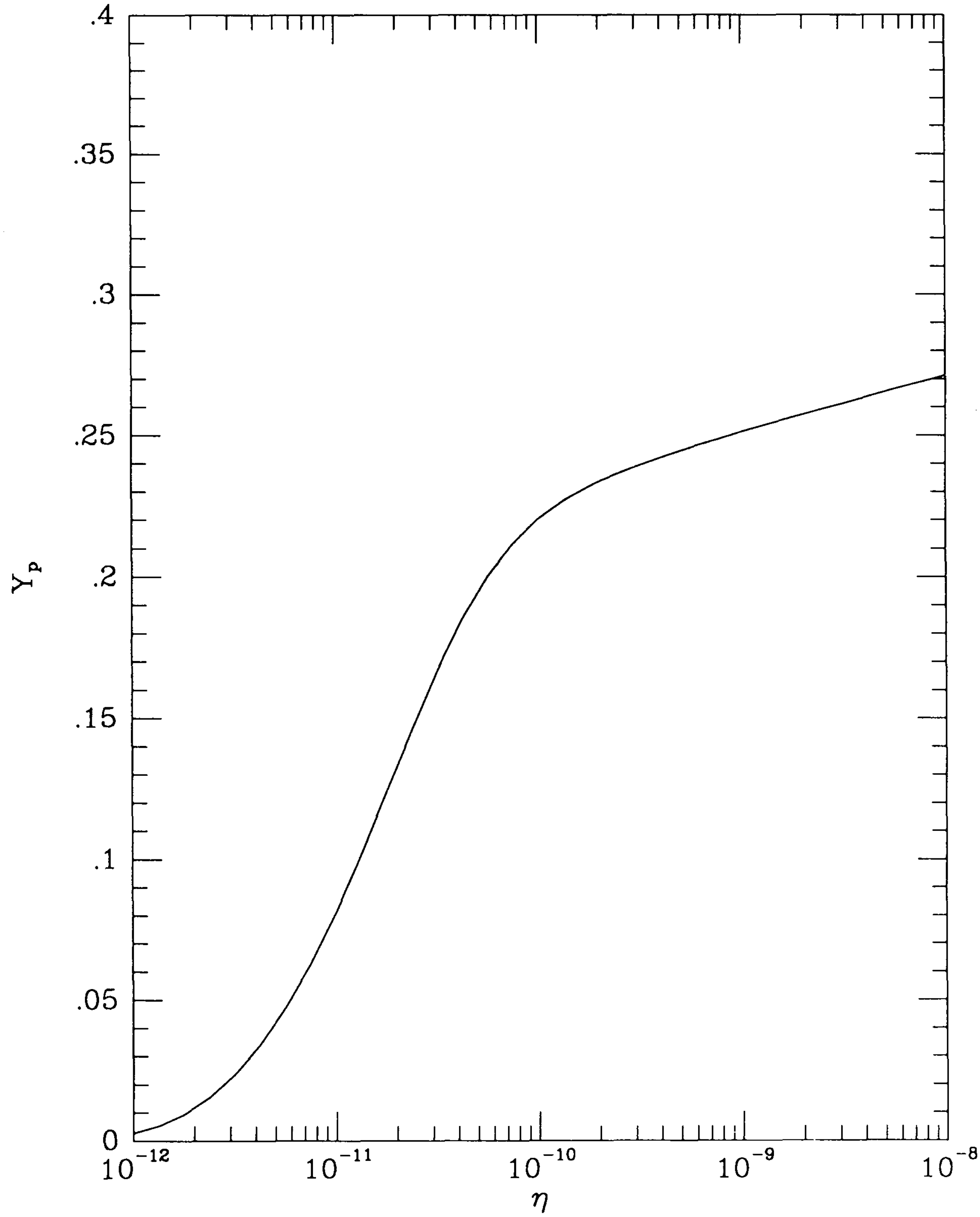
Recent reviews: Olive (1999), Tytler et al. (2000).

Detailed Computations, II



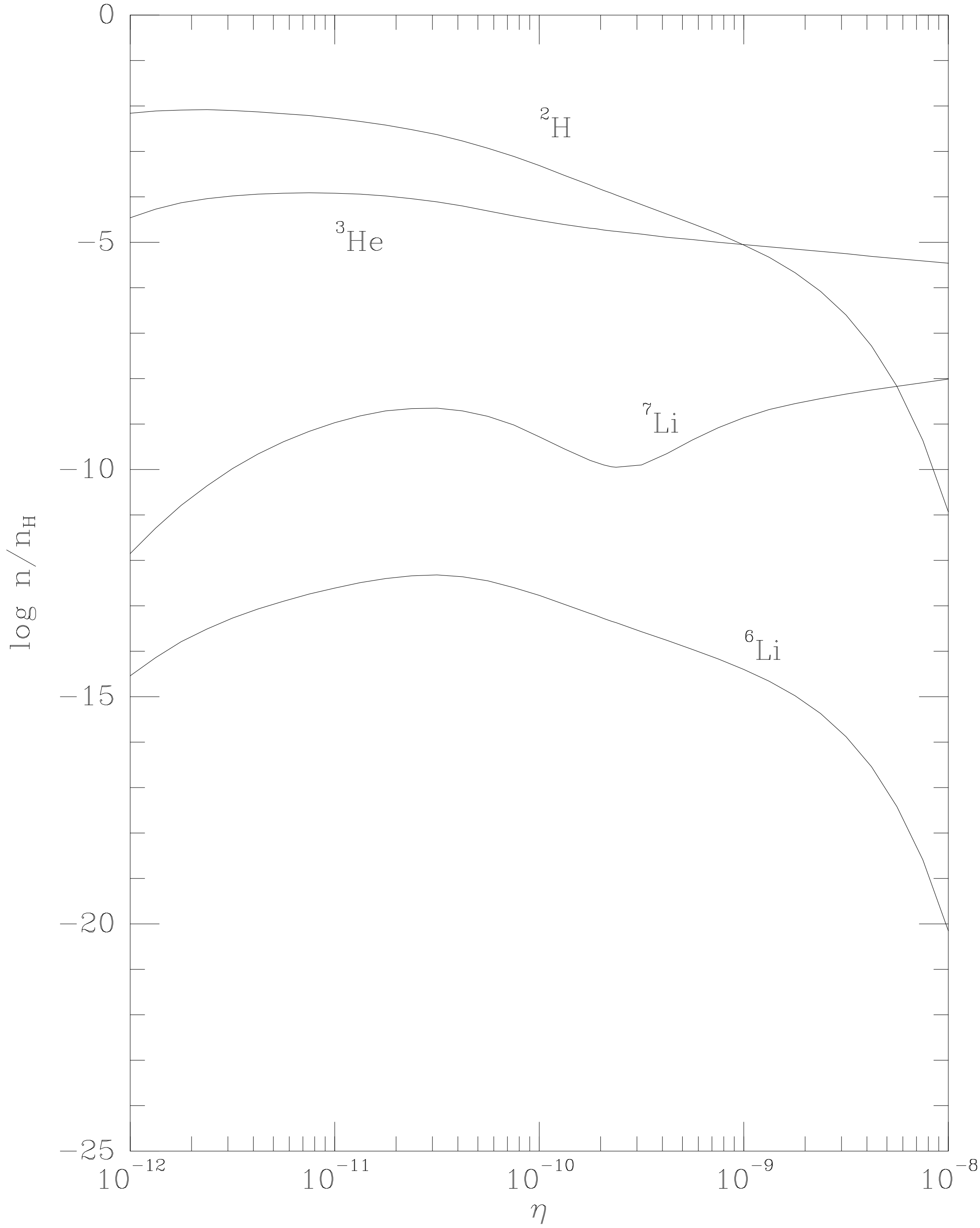
Build-up of abundances as function of time for $\eta = 5.1 \times 10^{-10}$ (Burles, Nollett & Turner, 1999, Fig. 3) [remember: $\eta = n_{\text{CMBR}}/n_{\text{baryons}}$]

Detailed Computations, III



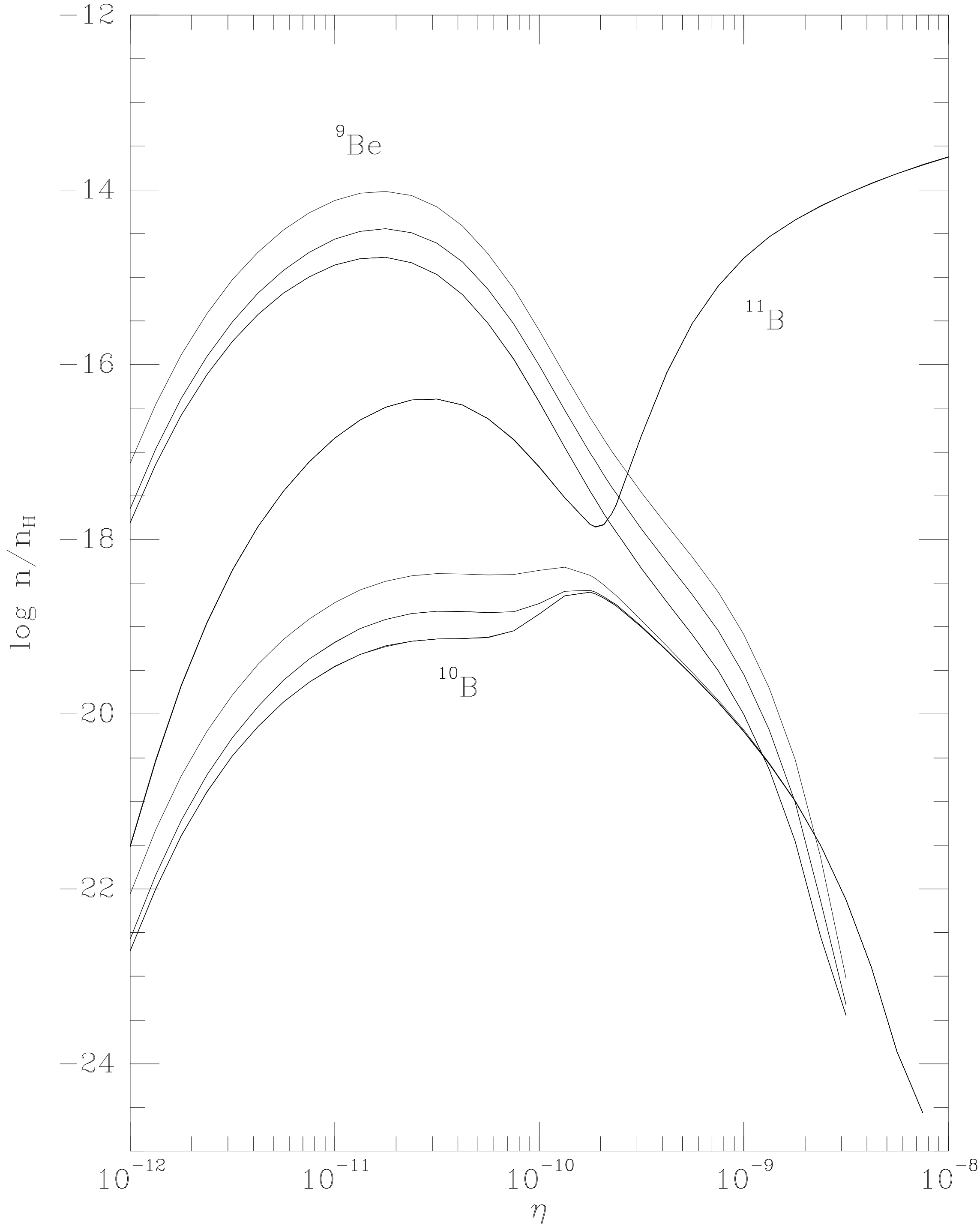
He abundance as function of η (Thomas et al., 1993, Fig. 3a).
^{4}He mainly dependent on

Detailed Computations, IV



Light-element abundances as function of η (Olive, 1999, Fig. 4)

Detailed Computations, V



Intermediate mass abundances as function of η (Olive, 1999, Fig. 5).

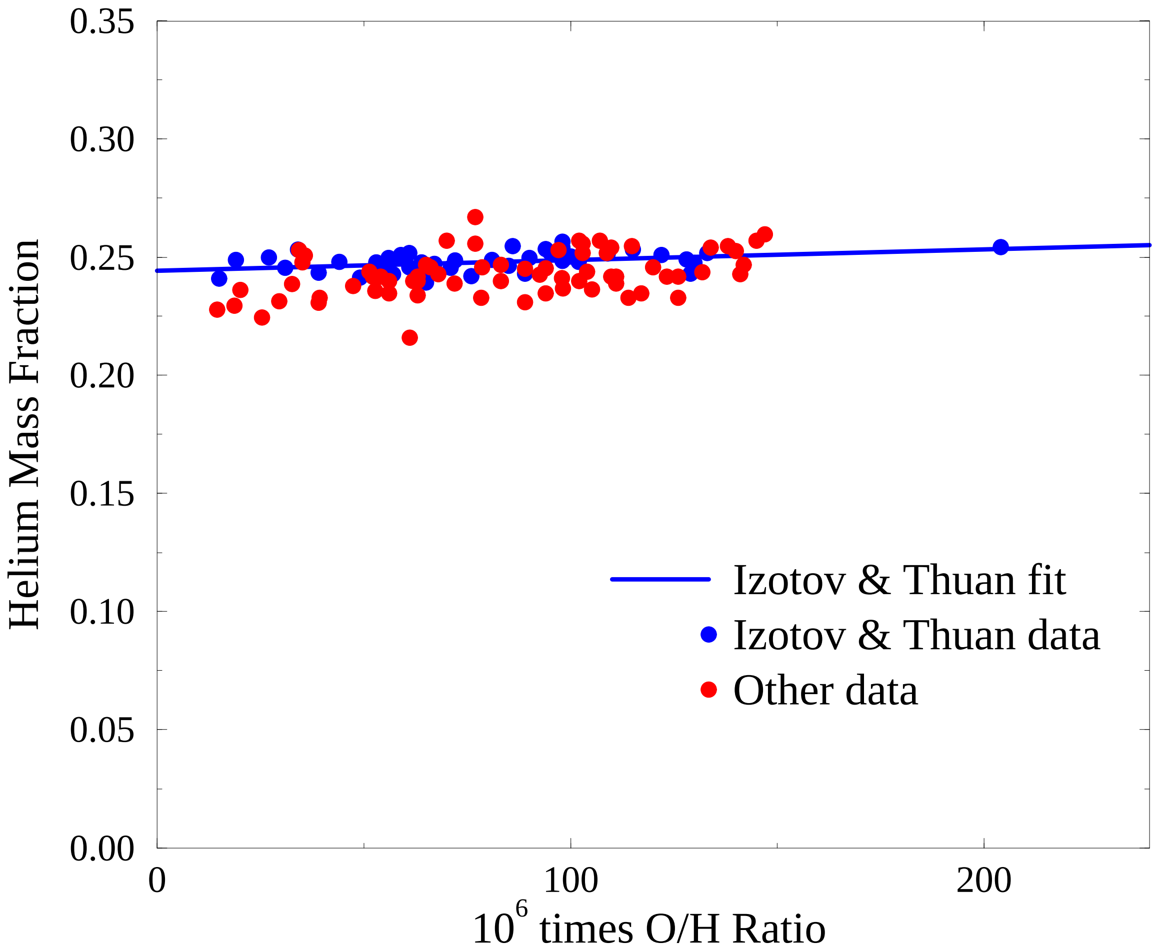
Remarkable Things

Note the following coincidences:

1. Freeze out of nucleons simultaneous to freeze out of neutrinos.
2. . . . and parallel to electron-positron annihilation.
3. Expansion slow enough that neutrons can be bound to nuclei.

$\Longrightarrow$ Long chain of coincidences makes our current universe possible!

^{4}He



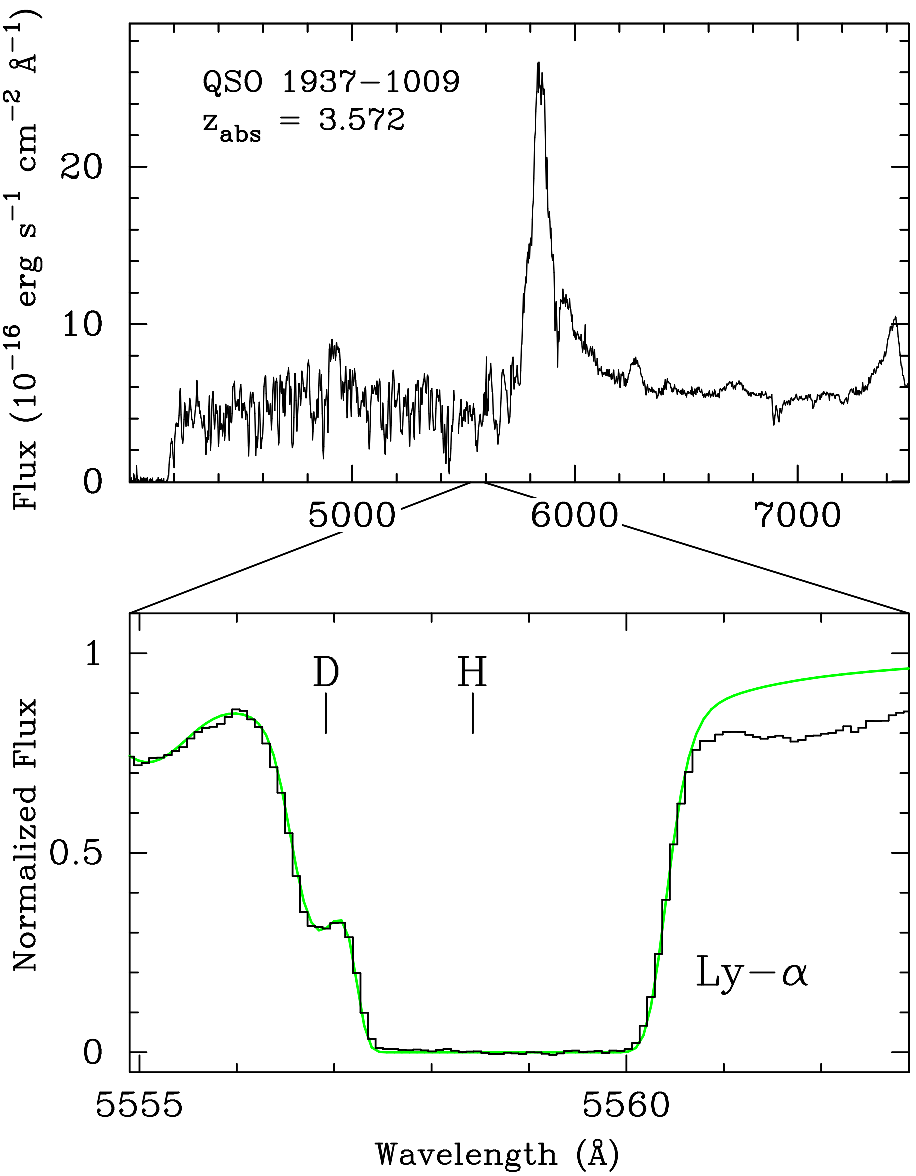
(Burles, Nollett & Turner, 1999, Fig. 4)

^{4}He produced in stars $\Longrightarrow$ extrapolate to zero metallicity in systems of low metallicity (i.e., minimize stellar processing).

Best determination from He II$\longrightarrow$He I recombination lines in H II regions (metallicity $\sim$ 20% solar).

Result: Linear correlation He vs. O
$\Longrightarrow$ extrapolate to zero oxygen to obtain primordial abundances. Result: $Y = 0.234 \pm 0.005$ (Olive, 1999).

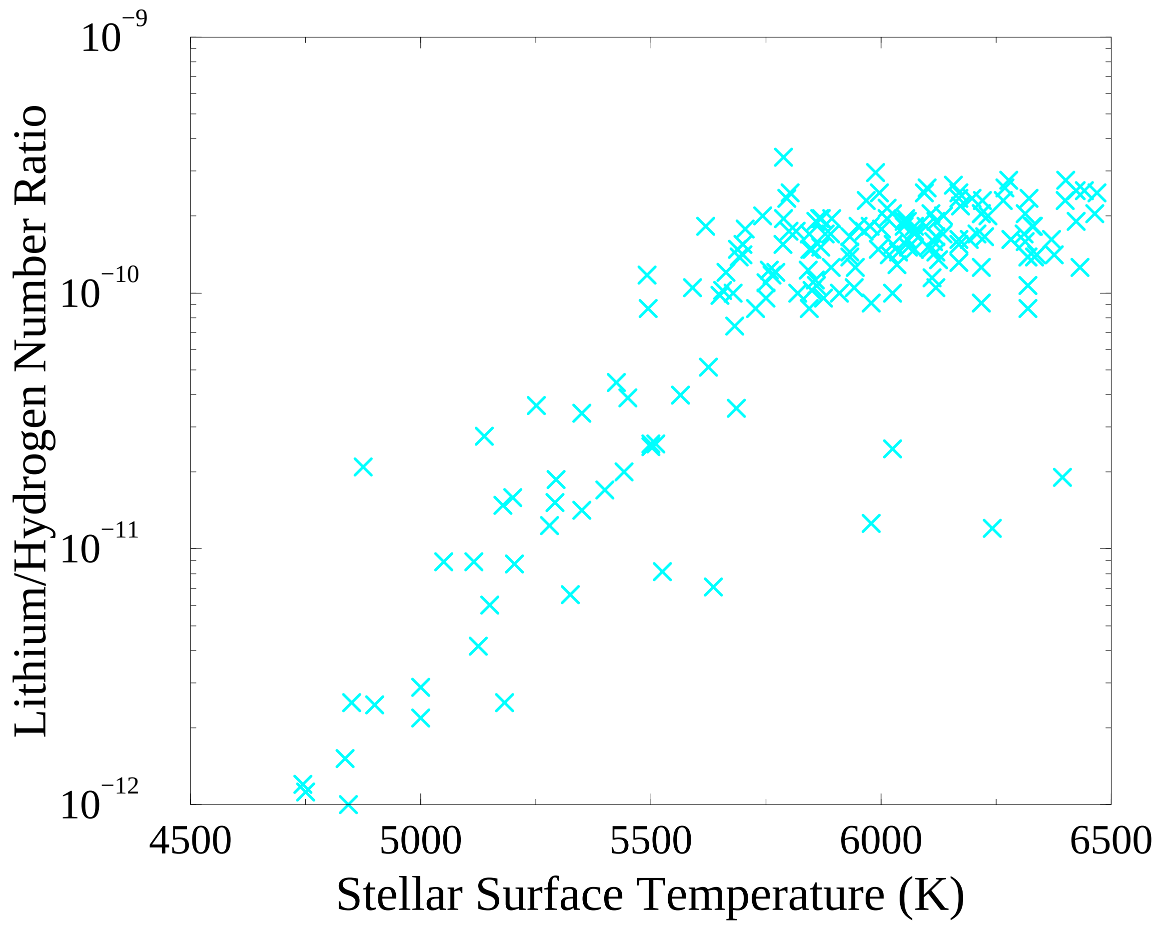
Deuterium



(Quasar 1937−1009; top: 3 m Lick, bottom: Keck; Burles, Nollett & Turner, 1999, Fig. 2)

Stars destroy D $\Longrightarrow$ use as non-processed material as possible!
Lyα forest (absorption of quasar light by intervening material)
$\Longrightarrow$ Structure caused by primordial deuterium, analysis of spectrum gives $\mathrm{D}/\mathrm{H} = (3.3 \pm 0.3) \times 10^{-5}$ (by number). Currently best measurement of primordial D-abundance.

Lithium



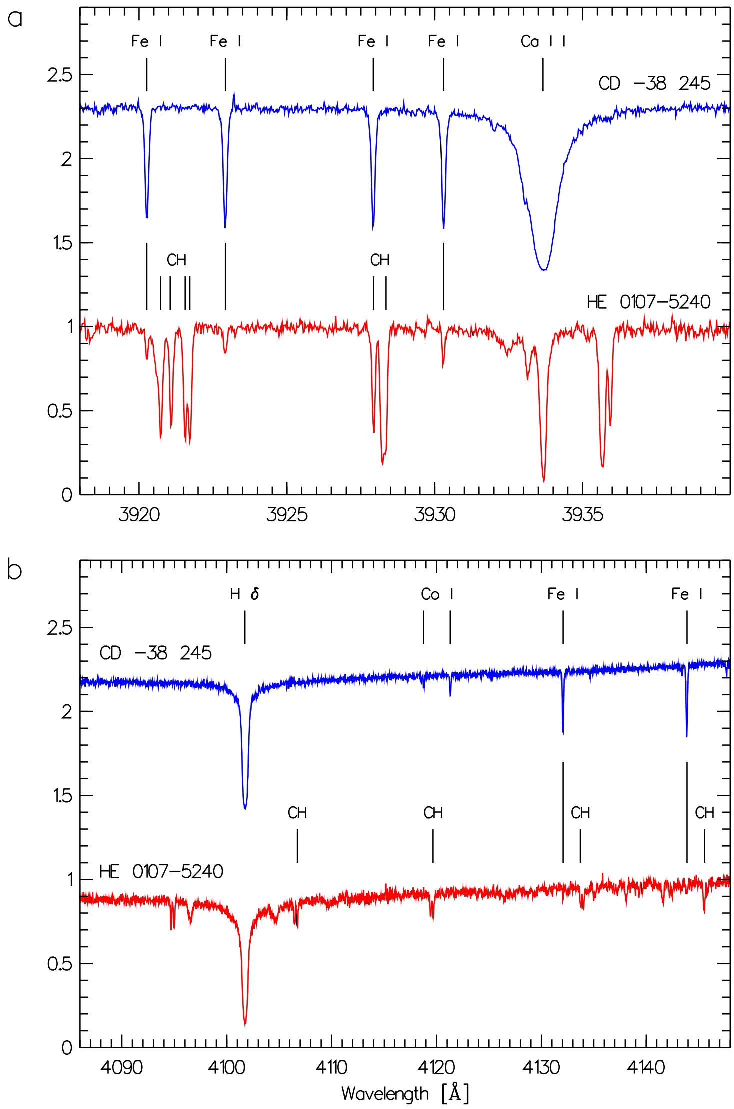
(Burles, Nollett & Turner, 1999, Fig. 5)

Stars with very low metallicity (old halo stars) show same Lithium abundance, ${}^7\mathrm{Li}/\mathrm{H} = 1.6 \times 10^{-10} \Longrightarrow$ close to primordial.

Cannot use galactic objects since Li also produced by spallation of heavier nuclei by cosmic rays (10× primordial produced this way).

Lower temperature stars: outer convection zone $\Longrightarrow$ Li burning destroys Li.

Population III



(after Christlieb et al., 2002, Fig. 1)

Earliest stars should only have H, He, i.e., $Z = 0$

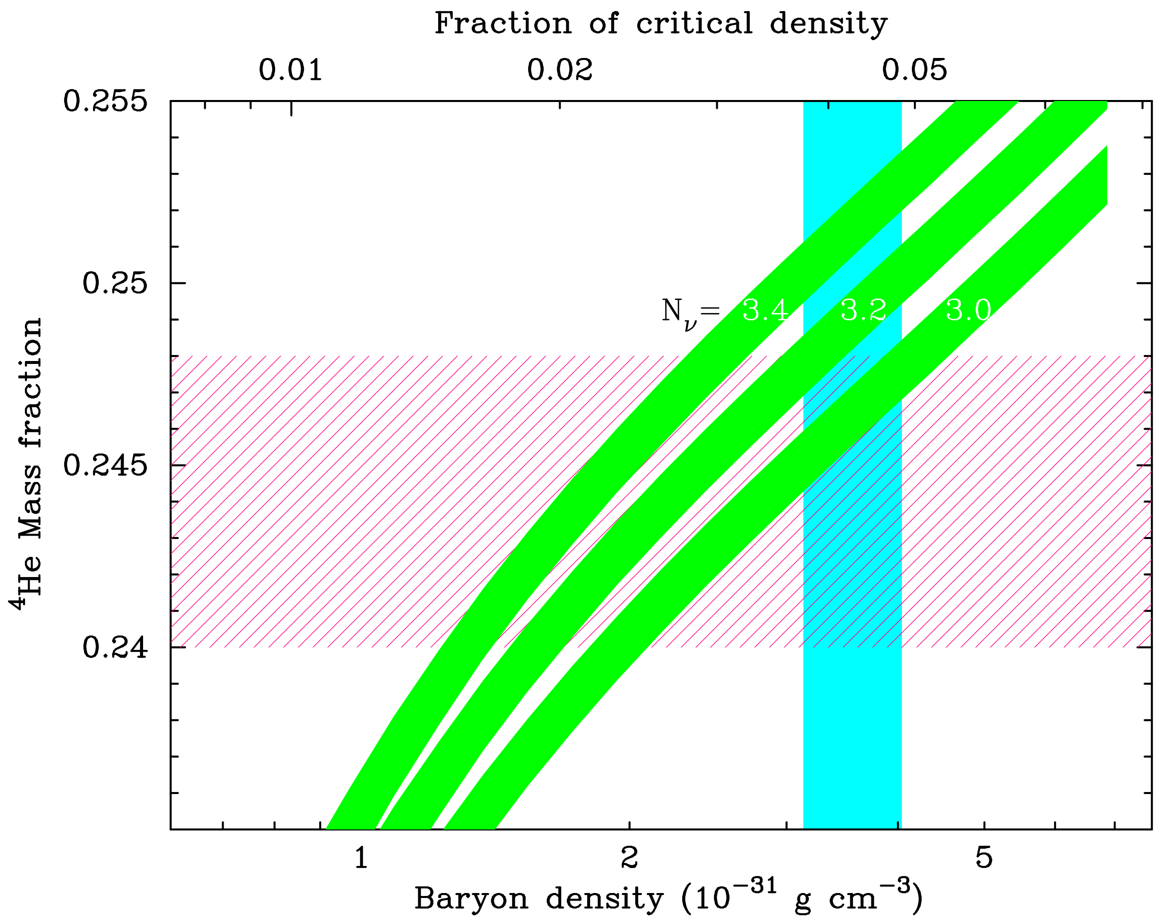
$\Longrightarrow$ would enable *direct* measure of primordial abundances.

Lowest metallicity known: HE0107−5240, with

Fe-abundance of 1/200000 solar

$\Longrightarrow$ “population III star”, formed either from primordial gas cloud (and got some elements later through accretion from ISM), or from debris from type II SN explosion.

Neutrino Species

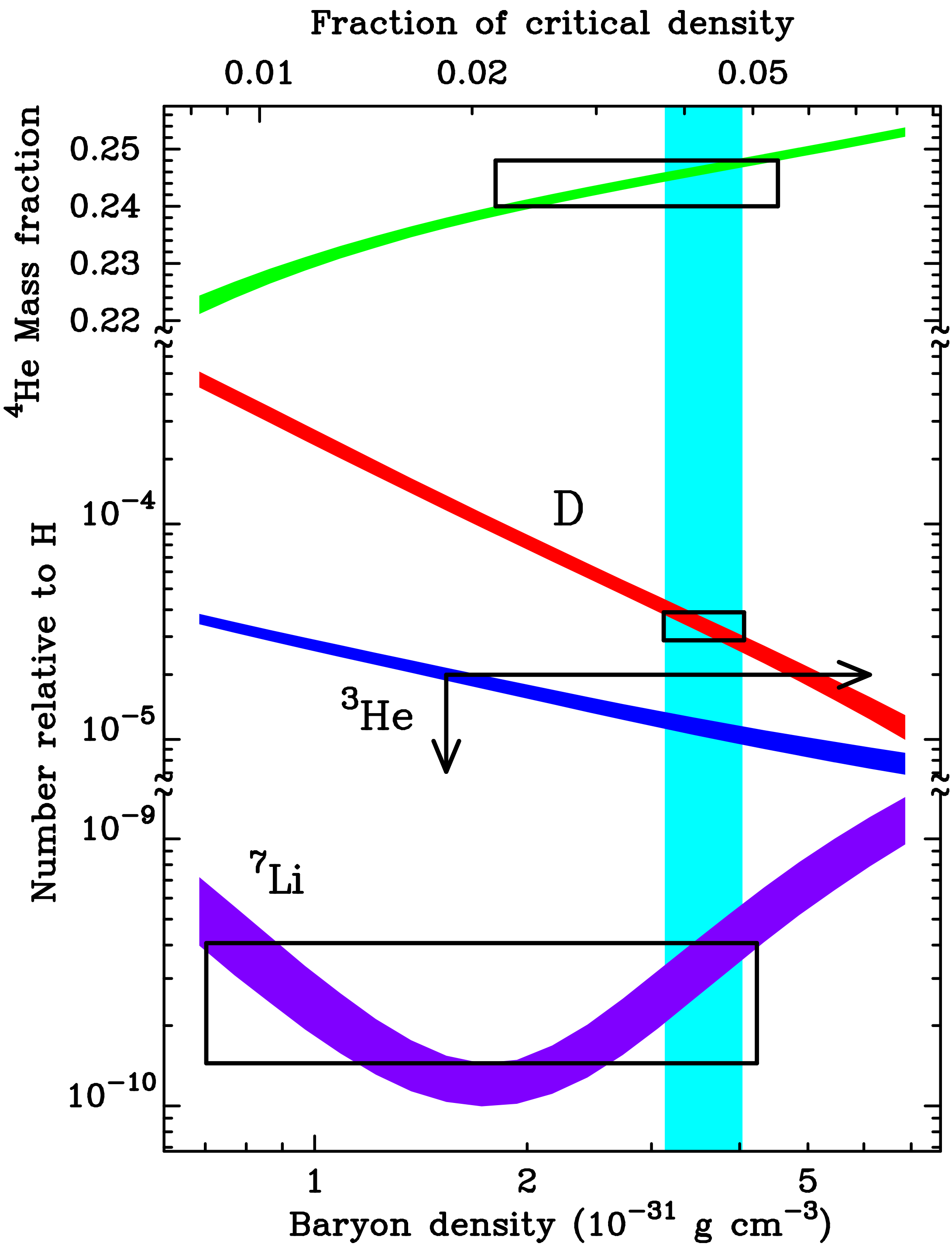


(Burles, Nollett & Turner, 1999, Fig. 7)

Number of neutrino species enters $\Omega \Longrightarrow$ Models for BBN constrain number of neutrino species to $N_\nu = 3$.

For a long time, BBN provided harder constraints on N_ν than laboratory experiments.

Summary



(Burles, Nollett & Turner, 1999, Fig. 1)

BBN strongly constrains Ω_{Baryons}.

Summary

Summary: History of the universe after its first 0.01 s (after Islam, 1992, Ch. 7, see also Weinberg, The first three minutes).

$t = 0.01\,\mathbf{s}$ $T = 10^{11}\,\mathbf{K}$ $\rho \sim 4 \times 10^{11}\,\mathbf{g\,cm^{-3}}$

Main constitutents: γ, ν, $\bar{\nu}$, e^--e^+ pairs.

No nuclei (instable). n and p in thermal balance.

$t = 0.1\,\mathbf{s}$ $T = 3 \times 10^{10}\,\mathbf{K}$ $\rho \sim 3 \times 10^{7}\,\mathbf{g\,cm^{-3}}$

Main constitutents: γ, ν, $\bar{\nu}$, e^--e^+ pairs. No nuclei.

$n + \nu \leftrightarrow p + e^-$: mass difference becomes important, 40% n, 60% p (by mass).

$t = 1.1\,\mathbf{s}$ $T = 10^{10}\,\mathbf{K}$ $\rho \sim 10^{5}\,\mathbf{g\,cm^{-3}}$

Neutrinos decouple, e^--e^+ pairs start to annihilate. No nuclei.

25% n, 75% p

Summary

$t = 13\,\mathbf{s}$ $\quad T = 3 \times 10^9\,\mathbf{K}$ $\quad \rho \sim 10^5\,\mathbf{g\,cm}^{-3}$

Reheating of photons, pairs annihilate, ν fully decoupled, deuterium still cannot form.

17% n, 83% p

$t = 3\,\mathbf{min}$ $\quad T = 10^9\,\mathbf{K}$ $\quad \rho \sim 10^5\,\mathbf{g\,cm}^{-3}$

Pairs are gone, neutron decay becomes important, start of nucleosynthesis

14% n, 86% p

$t = 35\,\mathbf{min}$ $\quad T = 3 \times 10^8\,\mathbf{K}$ $\quad \rho \sim 0.1\,\mathbf{g\,cm}^{-3}$

game over

Next important event: $t \sim 300000$ years:
Interaction CMB/matter stops ("last scattering", recombination).

Before we look at this, we look at

the first 0.01 s: the very early universe

UWarwick

Inflation

So far, have seen that BB works remarkably well in explaining the observed universe.

There are, however, quite big problems with the classical BB theories:

Horizon problem: CMB looks too isotropic $\Longrightarrow$ Why?

Flatness problem: Density close to BB was very close to $\Omega = 1$ (deviation $\sim 10^{-16}$ during nucleosynthesis) $\Longrightarrow$ Why?

Hidden relics problem: There are no observed magnetic monopoles, although predicted by GUT, neither gravitinos and other exotic particles $\Longrightarrow$ Why?

Vacuum energy problem: Energy density of vacuum is 10^{120} times smaller than predicted $\Longrightarrow$ Why?

Expansion problem: The universe expands $\Longrightarrow$ Why?

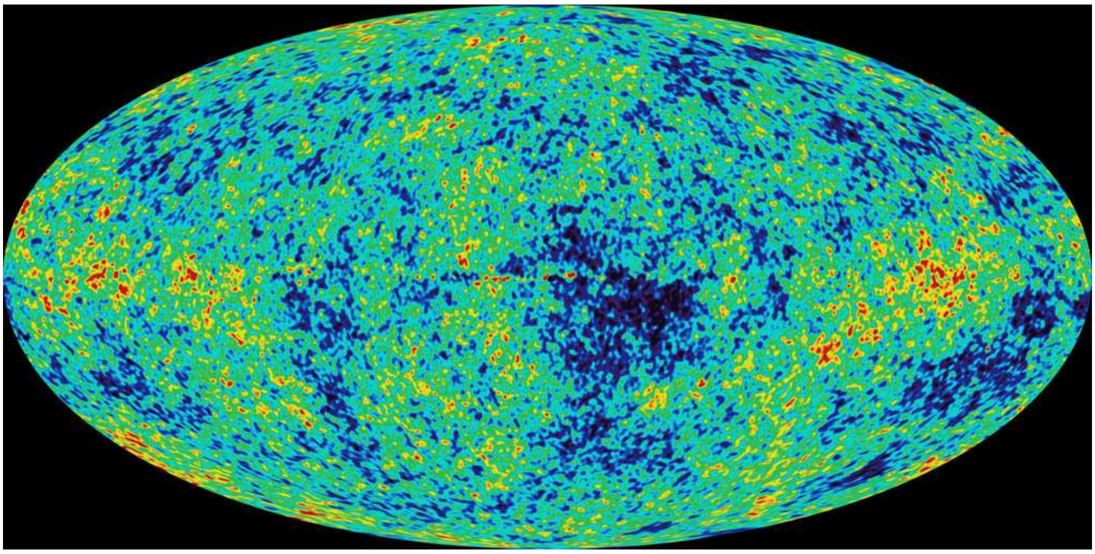
Baryogenesis: There is virtually no antimatter in the universe $\Longrightarrow$ Why?

Structure formation: Standard BB theory produces no explanation for lumpiness of universe.

Inflation attempts to answer all of these questions.

Recent Book: Liddle & Lyth (2000)

Horizon problem, I



(Bennett et al., 2003, temperature difference $\pm 200\,\mu$K)

COBE and WMAP: Temperature fluctuations in CMB on 10° scales:

$$\frac{\Delta T_{\mathrm{CMB}}}{T_{\mathrm{CMB}}} \sim 2 \times 10^{-5} \tag{6.107}$$

This is too small: Size of observable universe at given epoch ("particle horizon") is given by coordinate distance photons traveled since big bang (Eq. 4.46):

$$d_{\mathrm{h}} = R_0 \cdot r_{\mathrm{H}}(t) = \int_0^t \frac{c\,\mathrm{d}t}{a(t)} \tag{6.108}$$

For a matter dominated universe with $\Omega = 1$,

$$a(t) = \left(\frac{3H_0}{2}\,t\right)^{2/3} \tag{4.77}$$

such that for $t = t_0 = 2/(3H_0)$ (Eq. 4.78):

$$d_{\mathrm{h}}(t_0) = \frac{3c}{(3H_0/2)^{2/3}}\,t_0^{1/3} = \frac{2c}{H_0} \tag{6.109}$$

Horizon problem, II

For matter dominated universes at redshift z, Eq. (6.109) works out to be (Peacock, 1999, eq. 11.2):

$$d_\mathrm{h} \approx \frac{6000}{\sqrt{\Omega z}}\, h^{-1}\,\mathrm{Mpc} \tag{6.110}$$

CMB decoupled from matter at $z \sim 1000$ (see later), such that then $d_\mathrm{h} \sim 200\,\mathrm{Mpc}$, while today $d_\mathrm{h} \sim 6000\,\mathrm{Mpc} \Longrightarrow$ current observable volume $\sim 30000\times$ larger!

Note: we use $a \Longrightarrow$ all scales refer to what they are *now*, not what they *were* when the photons started!

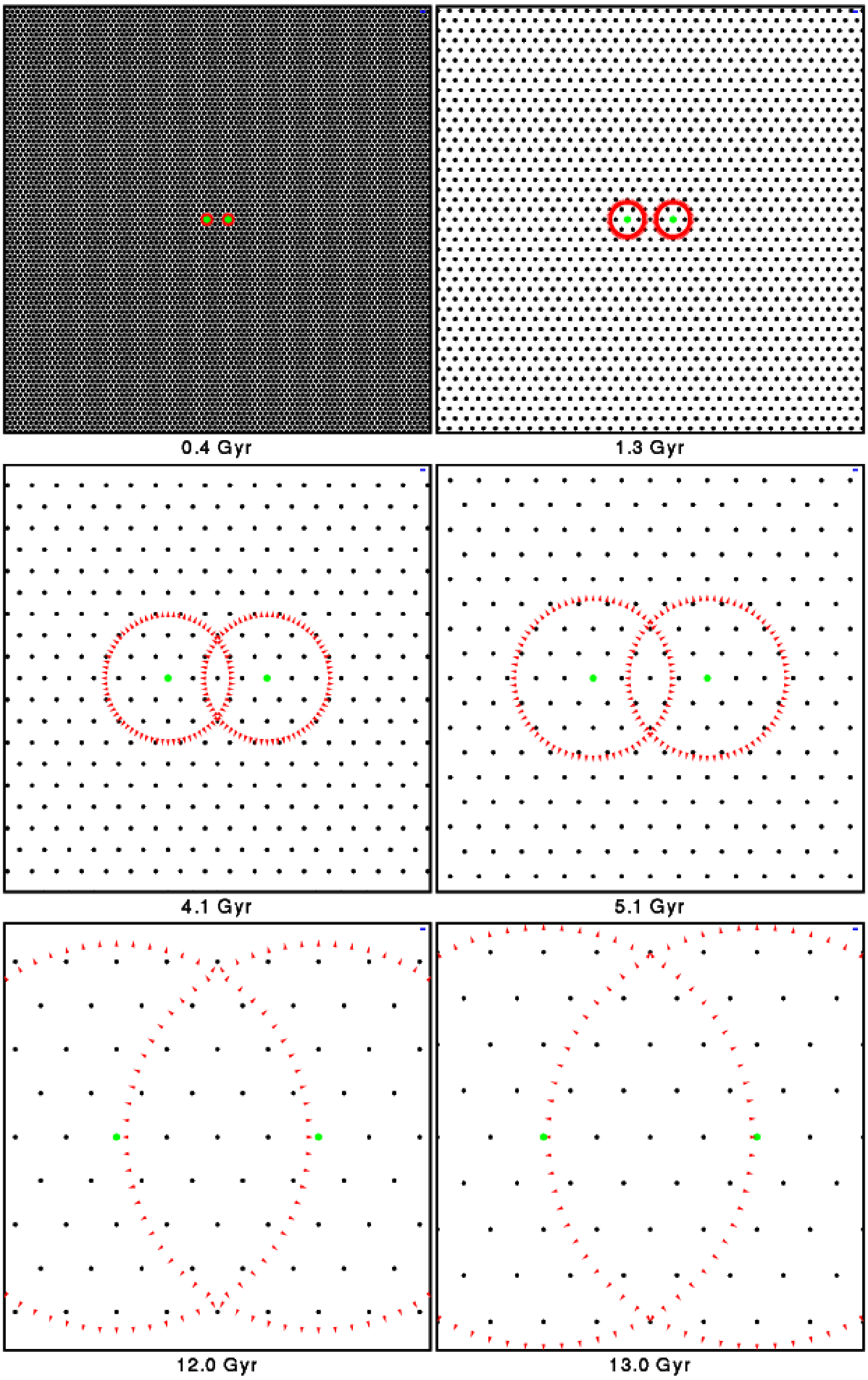
Horizon problem: Why were causally disconnected areas on the sky so similar when CMB last interacted with matter?

Note that the horizon distance is larger than Hubble length:

$$d_\mathrm{h} = \frac{2c}{H_0} > \frac{2c}{3H_0} = c \cdot t_0 = d_\mathrm{H} \tag{6.111}$$

Reason for this is that universe expanded while photons traveled towards us $\Longrightarrow$ Current observable volume larger than volume expected in a non-expanding universe.

Horizon problem, III



courtesy E. Wright.

Expansion of horizon in an expanding universe.

Flatness problem, I

Current observations of density of universe roughly imply

$$0.01 \lesssim \Omega \lesssim 2 \quad \text{i.e., } \Omega \sim 1 \tag{6.112}$$

(will be better constrained later).

$\Omega \sim 1$ imposes very strict conditions on initial conditions of universe:

The Friedmann equation (e.g., Eq. 4.61) can be written in terms of Ω:

$$\Omega - 1 = \frac{k}{a^2 H^2} = \frac{ck}{\dot{a}^2} \tag{6.113}$$

For a nearly flat, matter dominated universe, $a(t) \propto t^{2/3}$, such that

$$\frac{\Omega(t) - 1}{\Omega(t_0) - 1} = \left(\frac{t}{t_0}\right)^{2/3} \tag{6.114}$$

while for the radiation dominated universe with $a(t) \propto t$,

$$\frac{\Omega(t) - 1}{\Omega(t_0) - 1} = \frac{t}{t_0} \tag{6.115}$$

Flatness problem, II

Today: $t_0 = 3.1 \times 10^{17}\,h^{-1}\,\mathrm{s}$, i.e., observed flatness predicts for era of nucleosynthesis ($t = 1\,\mathrm{s}$):

$$\frac{\Omega(1\,\mathrm{s}) - 1}{\Omega(t_0) - 1} \sim 10^{-12} \ldots 10^{-16} \tag{6.116}$$

i.e., very close to unity.

Flatness problem: It is very unlikely that Ω was so close to unity at the beginning without a physical reason.

Had Ω been different from 1, the universe would immediately have been collapsed or expanded too fast $\Longrightarrow$ Anthropocentric point of view *requires* $\Omega = 1$.

Hidden relics problem

Modern theories of particle physics predict the following particles to exist:

Gravitinos: From supergravity, spin $3/2$ particle with $mc^2 \sim 100\,\text{GeV}$, if it exists, then nucleosynthesis would not work if BB started at $kT > 10^9\,\text{GeV}$.

Moduli: Spin-0 particles from superstring theory, contents of vacuum at high energies.

Magnetic Monopoles: Predicted in grand unifying theories, but not observed.

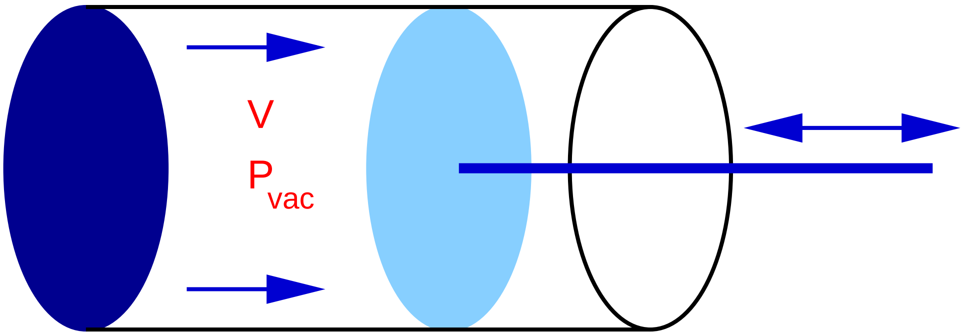
> Hidden relics problem: If there was a normal big bang, then strange particles should exist, which are not observed today.

Vacuum, Λ, I

What is vacuum? *Not empty space* but rather ground state of some physical theory.

Reviews: Carroll, Press & Turner (1992), Carroll (2001).

Since ground state should be same in all coordinate systems $\Longrightarrow$ Vacuum is Lorentz invariant.



(after Peacock, 1999, Fig. 1.3)

Equation of state (Zeldovich, 1968):

$$P_{\text{vac}} = -\rho_{\text{vac}} c^2 \tag{6.117}$$

This follows directly from 1st law of thermodynamics: ρ_{vac} should be constant if compressed or expanded, which is true only for this type of equation of state:

$$\mathrm{d}E = \mathrm{d}U + P\,\mathrm{d}V = \rho_{\text{vac}} c^2\,\mathrm{d}V - \rho_{\text{vac}} c^2\,\mathrm{d}V = 0 \tag{6.118}$$

An alternative derivation goes via the stress-energy momentum tensor of a perfect fluid, see Carroll, Press & Turner (1992).

Vacuum, Λ, II

ρ_{vac} defines Einstein's cosmological constant

$$\Lambda = -\frac{8\pi G \rho_{\text{vac}}}{c^4} \tag{6.119}$$

Adding ρ_{vac} to the Friedmann equations allows to define

$$\Omega_\Lambda = \frac{\rho_{\text{vac}}}{\rho_{\text{crit}}} = \frac{\rho_{\text{vac}}}{3H^2/8\pi G} = \frac{c^4\Lambda}{3H^2} \tag{6.120}$$

Classical physics: Particles have energy

$$E = T + V \tag{6.121}$$

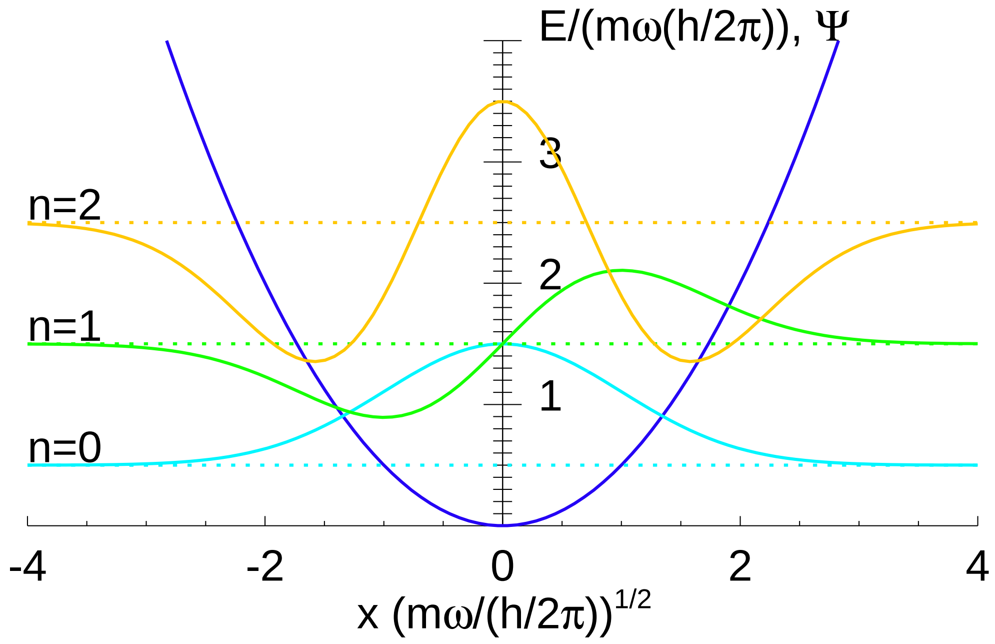
and force is $F = -\nabla V$, i.e., can add constant without changing equation of motion

$\Longrightarrow$ In classical physics, we are able to define $\rho_{\text{vac}} = 0$!

Quantum mechanics is (as usual) more difficult.

Vacuum, Λ, III

Vacuum in quantum mechanics:



Simplest case: harmonic oscillator:

$$V(x) = \frac{1}{2}m\omega^2x^2 \quad \text{i.e.,} \quad V(0) = 0 \tag{6.122}$$

However, particles can only have energies

$$E_n = \frac{1}{2}\hbar\omega + n\hbar\omega \quad \text{where } n \in \mathbb{N} \tag{6.123}$$

$\Longrightarrow$ Vacuum state has zero point energy

$$E_0 = \frac{1}{2}\hbar\omega \tag{6.124}$$

Simple consequence of uncertainty principle!

In QM, could normalize $V(x)$ such that $E_0 = 0$, important here is that vacuum state energy *differs* from classical expectation!

Vacuum, Λ, IV

Quantum field theory: Field as collection of harmonic oscillators of all frequencies. Simplest case: spinless boson ("scalar field", ϕ).

$\Longrightarrow$ Vacuum energy sum of all contributing modes:

$$E_0 = \sum_j \frac{1}{2}\hbar\omega_j \tag{6.125}$$

Compute sum by putting system in box with volume L^3, and then $L \longrightarrow \infty$.

Box $\Longrightarrow$ periodic boundary conditions:

$$\lambda_i = L/n_i \quad \Longleftrightarrow \quad k_i = 2\pi/\lambda_i = 2\pi n_i/L \tag{6.126}$$

for $n_i \in \mathbb{N} \Longrightarrow \mathrm{d}k_i L/2\pi$ discrete wavenumbers in $[k_i, k_i + \mathrm{d}k_i]$, such that

$$E_0 = \frac{1}{2}\hbar L^3 \int \frac{\omega_{\mathbf{k}}}{(2\pi)^3}\mathrm{d}^3\mathbf{k} \quad \text{where} \quad \omega_k^2 = k^2 + m^2/\hbar^2 \tag{6.127}$$

Imposing cutoff k_{max}:

$$\rho_{\text{vac}}c^2 = \lim_{L\to\infty} \frac{E_0}{L^3} = \hbar \frac{k_{\text{max}}^4}{16\pi^3} \tag{6.128}$$

Divergent for $k_{\text{max}} \longrightarrow \infty$ ("ultraviolet divergence").

Not worrisome: Expect QM to break down at large energies anyway (ignored collective effects, etc.).

Vacuum, Λ, V

When does classical quantum mechanics break down?

Estimate: Formation of "Quantum black holes":

$$\lambda_{\text{de Broglie}} = \frac{2\pi\hbar}{mc} < \frac{2Gm}{c^2} = r_{\text{Schwarzschild}} \tag{6.129}$$

$\Longrightarrow$ Defines Planck mass:

$$m_{\text{P}} = \sqrt{\frac{\hbar c}{G}} \mathrel{\widehat{=}} 1.22 \times 10^{19}\,\text{GeV} \tag{6.130}$$

Corresponding length scale: Planck length:

$$l_{\text{P}} = \frac{\hbar}{m_{\text{P}}} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-37}\,\text{cm} \tag{6.131}$$

...and time scale (Planck time):

$$t_{\text{P}} = \frac{l_{\text{P}}}{c} = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-47}\,\text{s} \tag{6.132}$$

$\Longrightarrow$ Limits of current physics until successful theory of quantum gravity.

The system of units based on l_{P}, m_{P}, t_{P} is called the system of Planck units.

Vacuum, Λ, VI

To compute QFT vacuum energy density, choose

$$k_{\text{max}} = m_{\text{P}}c^2/\hbar \tag{6.133}$$

Inserting into Eq. (6.128) gives

$$\rho_{\text{vac}}c^2 = 10^{74}\,\text{GeV}\,\hbar^{-3} \quad \text{or} \quad \rho_{\text{vac}} \sim 10^{92}\,\text{g}\,\text{cm}^{-3} \tag{6.134}$$

a tad bit on the high side ($\sim 10^{120}$ higher than observed).

Inserting ρ_{vac} in Friedmann equation:
$T < 3\,\text{K}$ at $t = 10^{-41}\,\text{s}$ after Big Bang.
To obtain current universe, require $k_{\text{max}} = 10^{-2}\,\text{eV} \Longrightarrow$ Less than binding energy of Hydrogen, where QM definitively works!

> Vacuum energy problem: Contributions from virtual fluctuations of all particles must cancel to very high precision to produce observable universe.

Casimir effect: force between conducting plates of area A and distance a in vacuum is $F_{\text{Casimir}} = \hbar c A \pi^2/(240a^4) \Longrightarrow$ caused by incomplete cancellation of quantum fluctuations. Confirmed by Lamoreaux in 1996 at 5% level.

Expansion problem

Cosmological Expansion:

GR predicts expansion of the universe, but initial conditions for expansion are not set!

Classical cosmology: "The unverse expands since it has expanded in the past"

$\Longrightarrow$ Hardly satisfying...

> Cosmological Expansion Problem: What is the physical mechanism responsible for the expansion of the universe?

To put it more bluntly:
"The Big Bang model explains nothing about the origin of the universe as we now perceive it, because all the most important features are 'predestined' by virtue of being built into the assumed initial conditions near to $t = 0$." (Peacock, 1999, p. 324)

Baryogenesis

Quantitatively: Today:

$$\frac{N_{\mathrm{p}}}{N_\gamma} \sim 10^{-9} \quad \text{but} \quad \frac{N_{\bar{\mathrm{p}}}}{N_\gamma} \sim 0 \tag{6.135}$$

Assuming isotropy and homogeneity, this is puzzling: Violation of Copernican principle!

Antimatter problem: There are more particles than antiparticles in the observable universe.

Sakharov (1968): Asymmetry implies three fundamental properties for theories of particle physics:

1. CP violation (particles and antiparticles must behave differently in reactions, observed, e.g., in the K^0 meson),
2. Baryon number violating processes (more baryons than antibaryons $\Longrightarrow$ Prediction by GUT),
3. Deviation from thermal equilibrium in early universe (CPT theorem: $m_{\mathrm{X}} = m_{\bar{\mathrm{X}}} \Longrightarrow$ *same* number of particles and antiparticles in thermal equilibrium).

Structure formation

Final problem: structure formation

In the classical BB picture, the initial conditions for structure formation observed are not explained. Furthermore, assuming the observed Ω_{baryons}, the observed structures (=us) cannot be explained.

The theory of inflation attempts to explain all of the problems mentioned by invoking phase of exponential expansion in the very early universe ($t \lesssim 10^{-16}$ s).

Basic Idea, I

Use the Friedmann equation with a cosmological constant:

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad (6.136)$$

Basic assumption of inflationary cosmology:

> During the big bang there was a phase where Λ dominated the Friedmann equation.

$$H(t) = \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} = \text{const.} \qquad (6.137)$$

since $\Lambda = \text{const.}$ (probably…).

Solution of Eq. (6.137):

$$a \propto \mathrm{e}^{Ht} \qquad (6.138)$$

and inserting into Eq. (6.113) shows that

$$\Omega - 1 = \frac{k}{a^2H^2} \propto \mathrm{e}^{-2Ht} \qquad (6.139)$$

Basic Idea, II

When did inflation happen?

Typical assumption: Inflation $=$ phase transition of a scalar field (“inflaton”) associated with Grand Unifying Theories.

Therefore the assumptions:

- temperature $kT_{\mathrm{GUT}} = 10^{15}\,\mathrm{GeV}$, when $1/H \sim 10^{-34}\,\mathrm{sec}$ ($t_{\mathrm{start}} \sim 10^{-34}\,\mathrm{sec}$).
- inflation lasted for 100 Hubble times, i.e., for $\Delta T = 10^{-32}\,\mathrm{sec}$.

With Eq. (6.138):

Inflation: Expansion by factor $\mathrm{e}^{100} \sim 10^{43}$.

. . . corresponding to a volume expansion by factor $\sim 10^{130} \Longrightarrow$ solves hidden relics problem!

Furthermore, Eq. (6.139) shows

$$\Omega - 1 = 10^{-86} \tag{6.140}$$

$\Longrightarrow$ solves flatness problem!

Basic Idea, III

Temperature behavior: *During* inflation universe supercools:

Remember: entropy density

$$s = \frac{\rho c^2 + P}{T} \tag{6.79}$$

But for Λ:

$$p = -\rho c^2 \tag{6.117}$$

so that the entropy density of vacuum

$$s_{\text{vac}} = 0 \tag{6.141}$$

Trivial result since vacuum is just one quantum state $\Longrightarrow$ *very* low entropy.

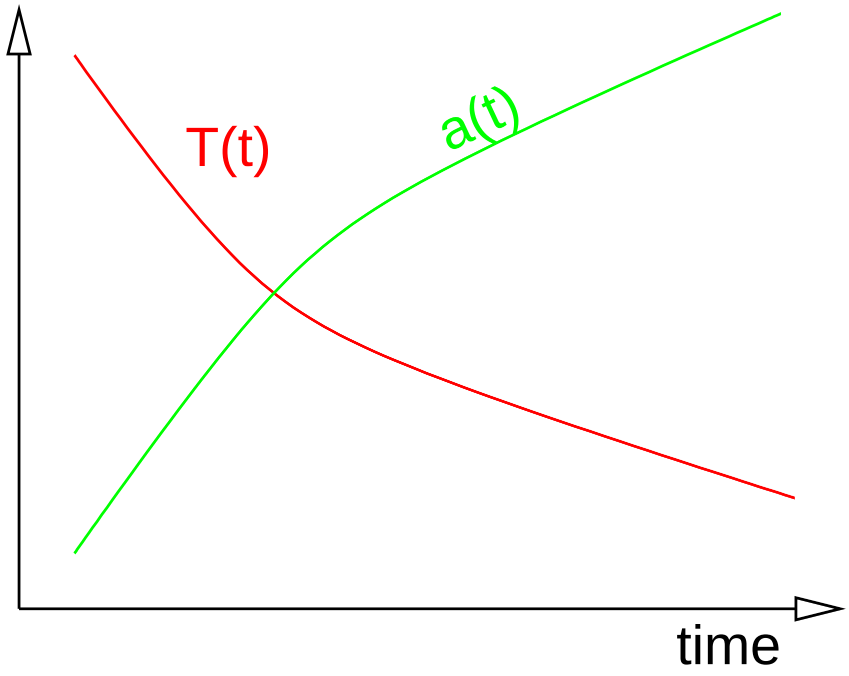
Inflation produces no entropy $\Longrightarrow$ S existing *before* inflation gets diluted, since entropy *density* $s \propto a^{-3}$.
But for relativistic particles $s \propto T^3$ (Eq. 6.82), such that

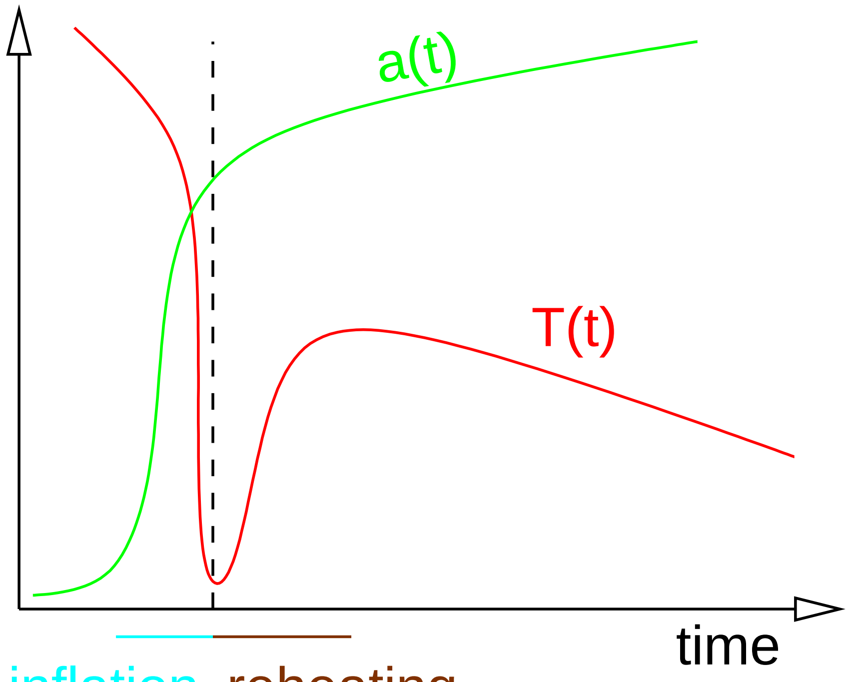
$$aT = \text{const.} \quad \Longrightarrow \quad T_{\text{after}} = 10^{-43} T_{\text{before}} \tag{6.142}$$

When inflation stops: vacuum energy of inflaton field transferred to normal matter
$\Longrightarrow$ "Reheating" to temperature

$$T_{\text{reheating}} \sim 10^{15}\,\text{GeV} \tag{6.143}$$

Summary





(after Bergström & Goobar, 1999, Fig. 9.1, and Kolb & Turner, Fig. 8.2)

Scalar Fields, I

For inflation to work: need short-term cosmological constant, i.e., need particles with negative pressure.

Basic idea (Guth, 1981): cosmological phase transition where suddenly a large Λ happens.

How? $\Longrightarrow$ Quantum Field Theory!

Describe hypothetical particle with a time-dependent quantum field, $\phi(t)$, and potential, $V(\phi)$.

Simplest example from QFT ($\hbar = c = 1$):

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \tag{6.144}$$

where m: "mass of field".

Particle described by ϕ: "inflaton".

For all scalar fields, particle physics shows:

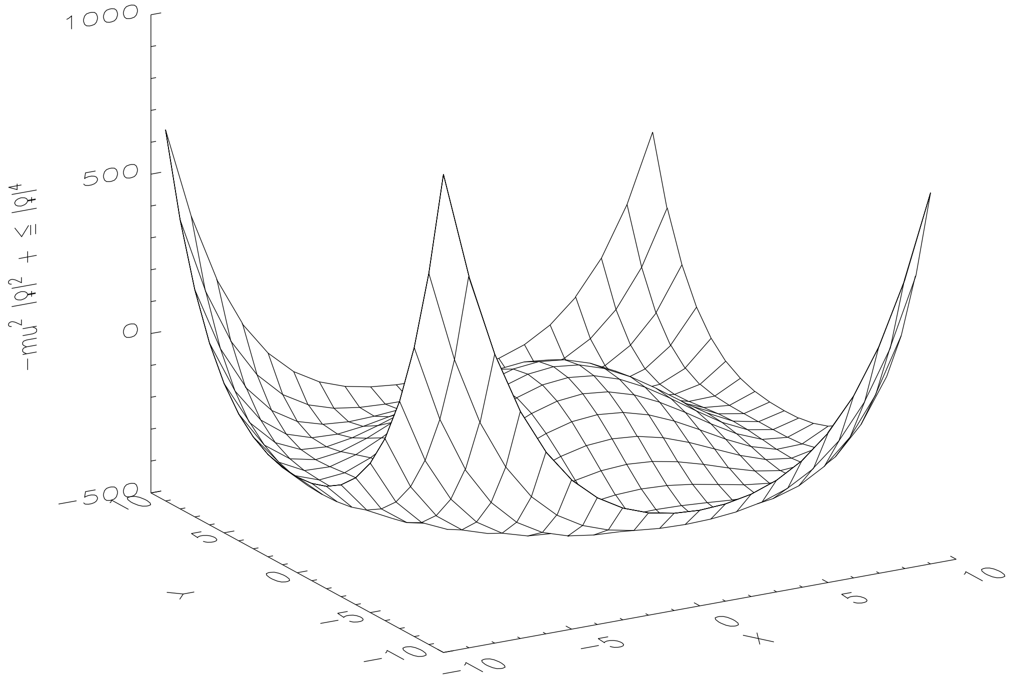
$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{6.145}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{6.146}$$

i.e., obeys vacuum EOS!

"Vacuum": particle "sits" at minimum of V.

Scalar Fields, II



Typically: potential looks more complicated.
Due to symmetry, after harmonic oscillator, 2nd simplest potential: Mexican hat potential ("Higgs potential"),

$$V(\phi) = -\mu^2\phi^2 + \lambda\phi^4 \qquad (6.147)$$

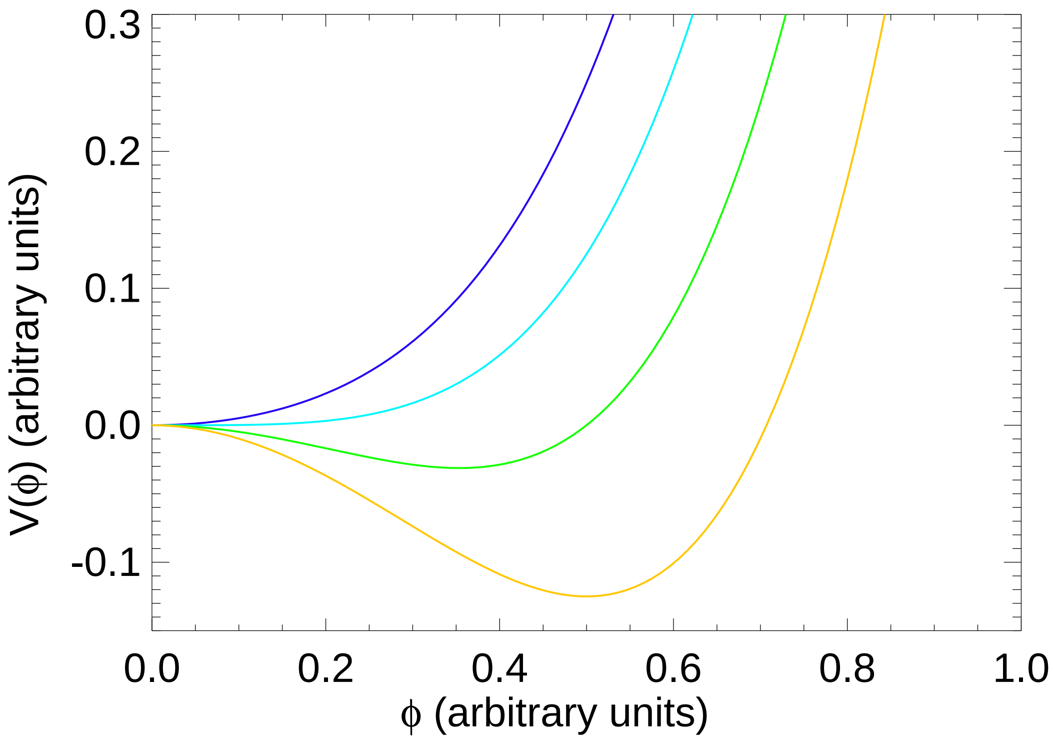
$\Longrightarrow$ Minimum of V still determines vacuum value.
For $T \neq 0$, need to take interaction with thermal bath into account $\Longrightarrow$ Temperature dependent potential!

$$V_{\text{eff}}(\phi) = -(\mu^2 - aT^2)\phi^2 + \lambda\phi^4 \qquad (6.148)$$

where a some constant.

(minimization of Helmholtz free energy, see Peacock, 1999, , p. 329ff., for details)

Scalar Fields, III



The minimum of V is at

$$\phi = \begin{cases} 0 & \text{for } T > T_\mathrm{c} \\ \sqrt{(\mu^2 - aT^2)/(2\lambda)} & \text{for } T < T_\mathrm{c} \end{cases} \tag{6.149}$$

where the critical temperature

$$T_\mathrm{c} = \mu/\sqrt{a} \tag{6.150}$$

and

$$V_\text{min} = \begin{cases} 0 & \text{for } T > T_\mathrm{c} \\ -\frac{(\mu^2 - aT^2)^2}{4\lambda} & \text{for } T < T_\mathrm{c} \end{cases} \tag{6.151}$$

Since switch happens suddenly: phase transition

Scalar Fields, IV

Minimum V_{min} for $T > T_{\text{c}}$ smaller than "vacuum minimum" $\Longrightarrow$ Behaves like a cosmological constant!

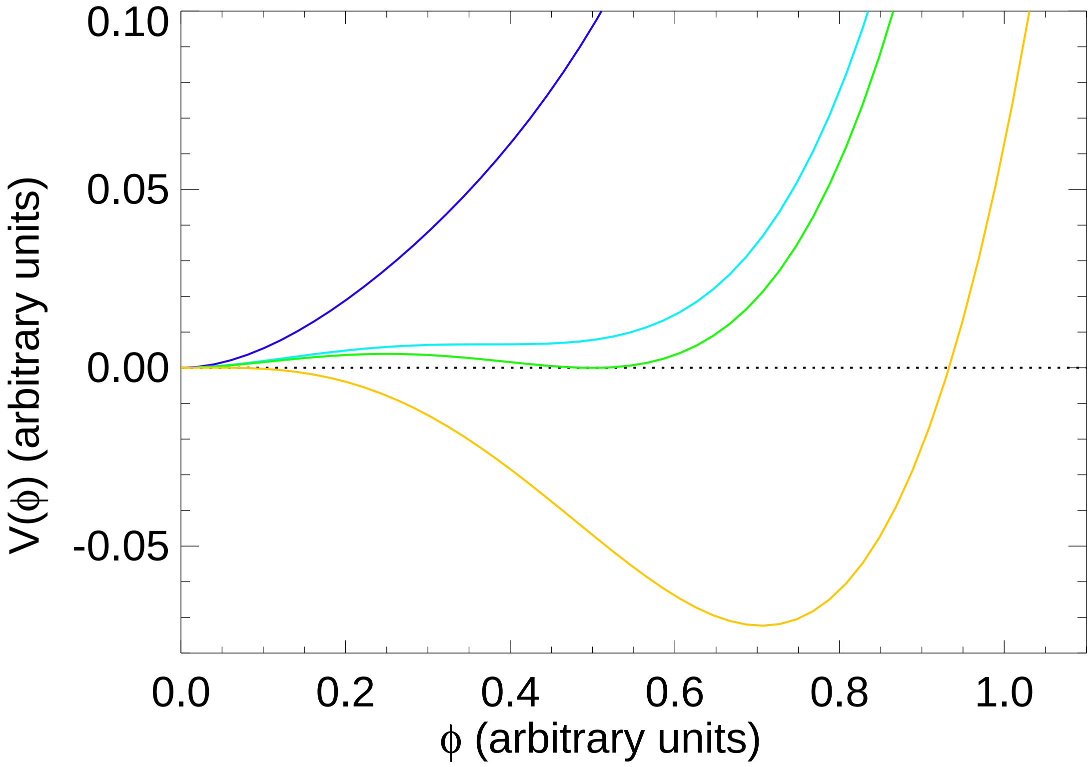
Since $T_{\text{c}} \propto \mu$,

> Inflation sets in at mass scale of whatever scalar field produces inflation.

Grand Unifying Theories: $m \sim 10^{15}$ GeV.

The problem is, what $V(\phi)$ to use...

First-Order Inflation



(after Peacock, 1999, Fig. 11.2)

Original idea (Guth, 1981):

$$V(\phi, T) = \lambda|\phi|^4 - b|\phi|^3 + aT^2|\phi|^2 \qquad (6.152)$$

has two minima for T greater than a critical temperature:

$V_{\text{min}}(\phi = 0)$: false vacuum

$V_{\text{min}}(\phi > 0)$: true vacuum iff < 0.

Particle can tunnel between both vacua: first order phase transition $\Longrightarrow$ first order inflation.

Problem: vacuum tunnels between false and true vacua $\Longrightarrow$ formation of bubbles.

Outside of bubbles: inflation goes infinitely ("graceful exit problem").

First order inflation is not feasible.

Summary

First order inflation does not work $\Longrightarrow$ Potentials derived from GUTs do not work.

$\Longrightarrow$ However, many empirical potentials do not suffer from these problems $\Longrightarrow$ inflation is *still* theory of choice.

Catchphrases (Liddle & Lyth, 2000, Ch. 8):

- supersymmetry/-gravitation $\Longrightarrow$ tree-level potentials,
- renormalizable global susy,
- chaotic inflation,
- power-law inflation,
- hybrid inflation (combination of *two* scalar fields) $\Longrightarrow$ spontaneous or dynamical susy breaking,
- scalar-tensor gravity

. . . and many more

All are somewhat *ad hoc*, and have more or less foundations in modern theories of QM and gravitation.

Information on what model correct from

1. predicted seed to structure formation, and
2. values of Ω and Λ.

$\Longrightarrow$ Determine Ω and Λ!

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