Introduction

Johannes Kepler: Motion of planets governed by three laws:

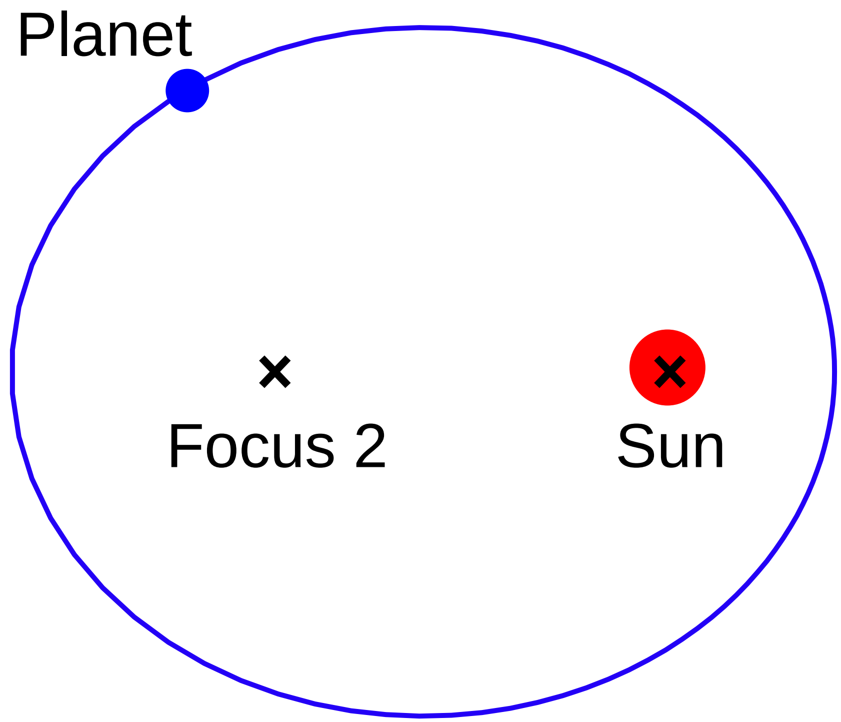
1. Each planet moves in an elliptical orbit, with the Sun at one focus of the ellipse. ("Astronomia Nova", 1609)
2. A line from the Sun to a given planet sweeps out equal areas in equal times. ("Astronomia Nova", 1609)
3. The square of the orbital periods of the planets is proportional to the cube of the major axes. ("Harmonice Mundi", 1619)

Isaac Newton ("Principia", 1687): Kepler's laws are consequence of gravitational interaction between planets and the Sun, and the gravitational force is

$$\boldsymbol{F}_1 = -\frac{Gm_1m_2}{r_{12}^2}\frac{\boldsymbol{r}_{21}}{r_{12}} \tag{4.1}$$

where $\boldsymbol{F}_1$ is the gravitational force exerted on object 1, m_1, m_2 are the masses of the interacting objects, r their distance, and $\boldsymbol{r}_{21}/r_{12}$ the unit vector joining the objects, $\boldsymbol{r}_{21} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, $\boldsymbol{r}_{12} = -\boldsymbol{r}_{21}$ and $r_{12} = |\boldsymbol{r}_{12}| = |\boldsymbol{r}_{21}|$.

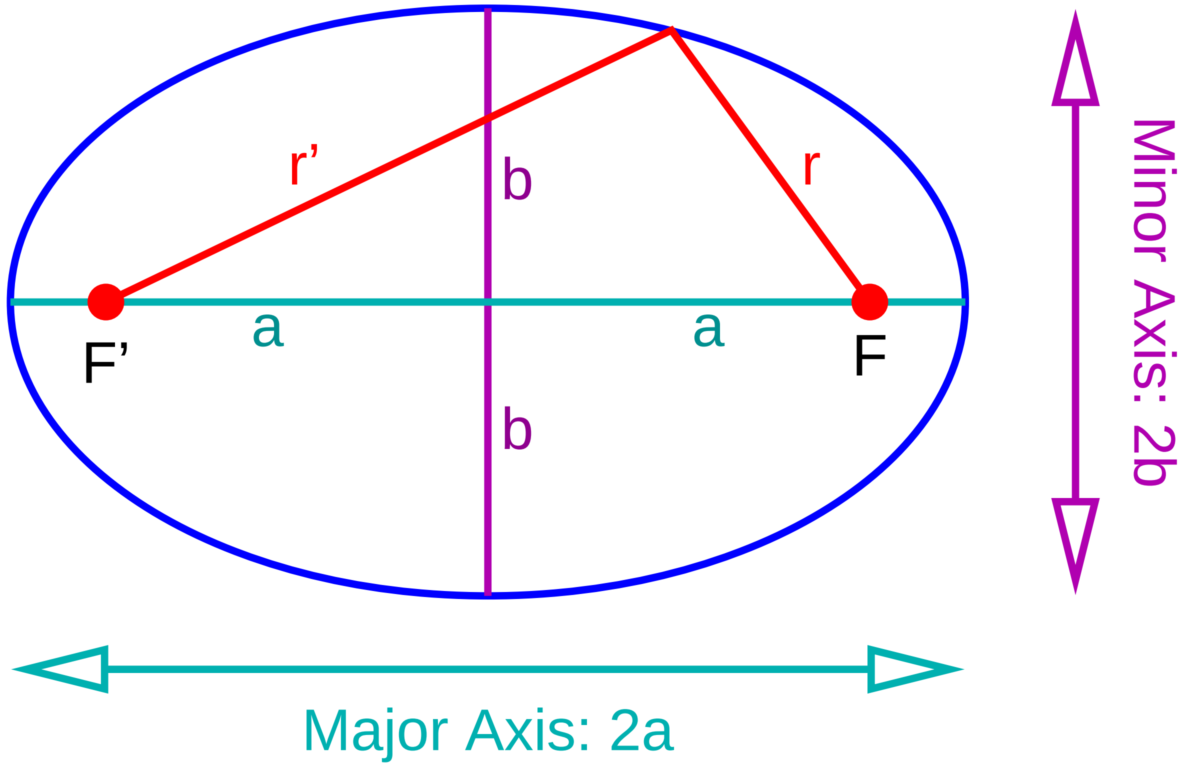
Keplers 1st Law



Kepler's 1st Law: The orbits of the planets are ellipses and the Sun is at one focus of the ellipse.

For the planets of the solar system, the ellipses are almost circular, for comets they can be very eccentric.

Keplers 1st Law

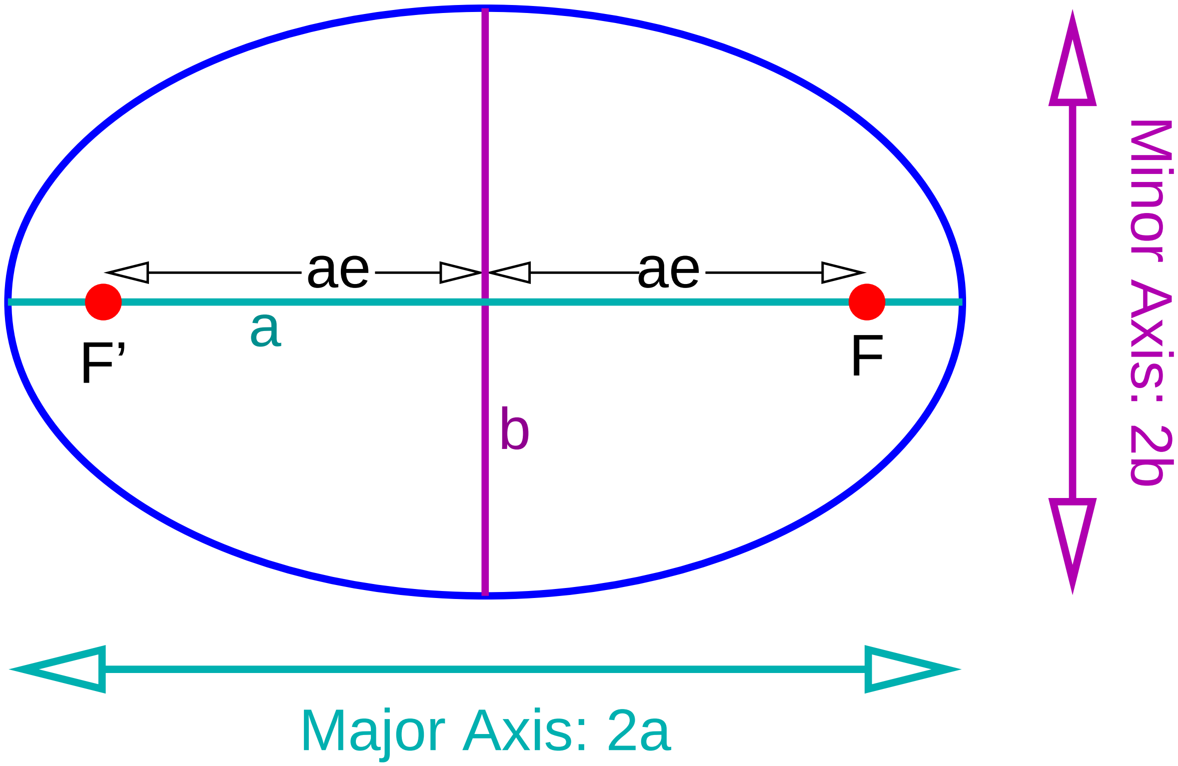


Definition: Ellipse = Sum of distances r, r' from any point on ellipse to two fixed points (foci, singular: focus), F, F', is constant:

$$r + r' = 2a \tag{4.2}$$

where a is called the semi-major axis of the ellipse.

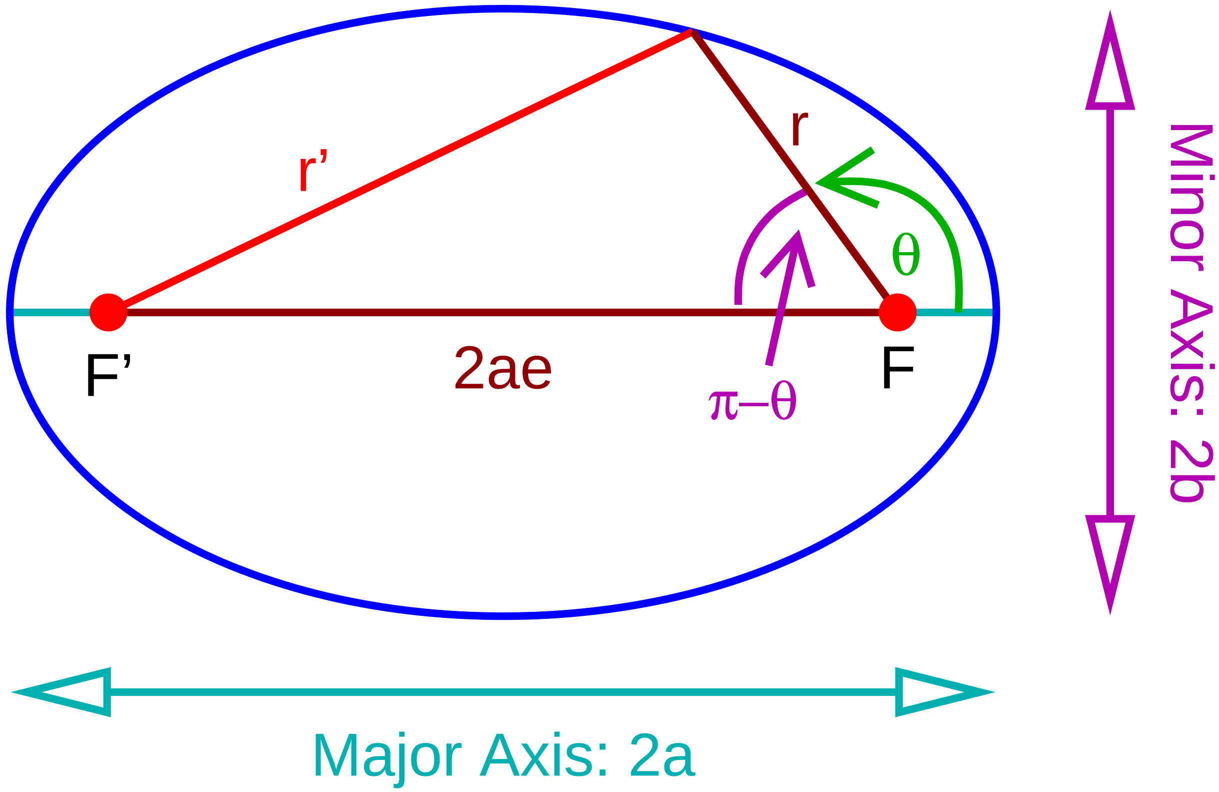
Keplers 1st Law



Definition: Eccentricity e: ratio between distance from centre of ellipse to focal point and semi-major axis.

So circles have $e = 0$.

Keplers 1st Law



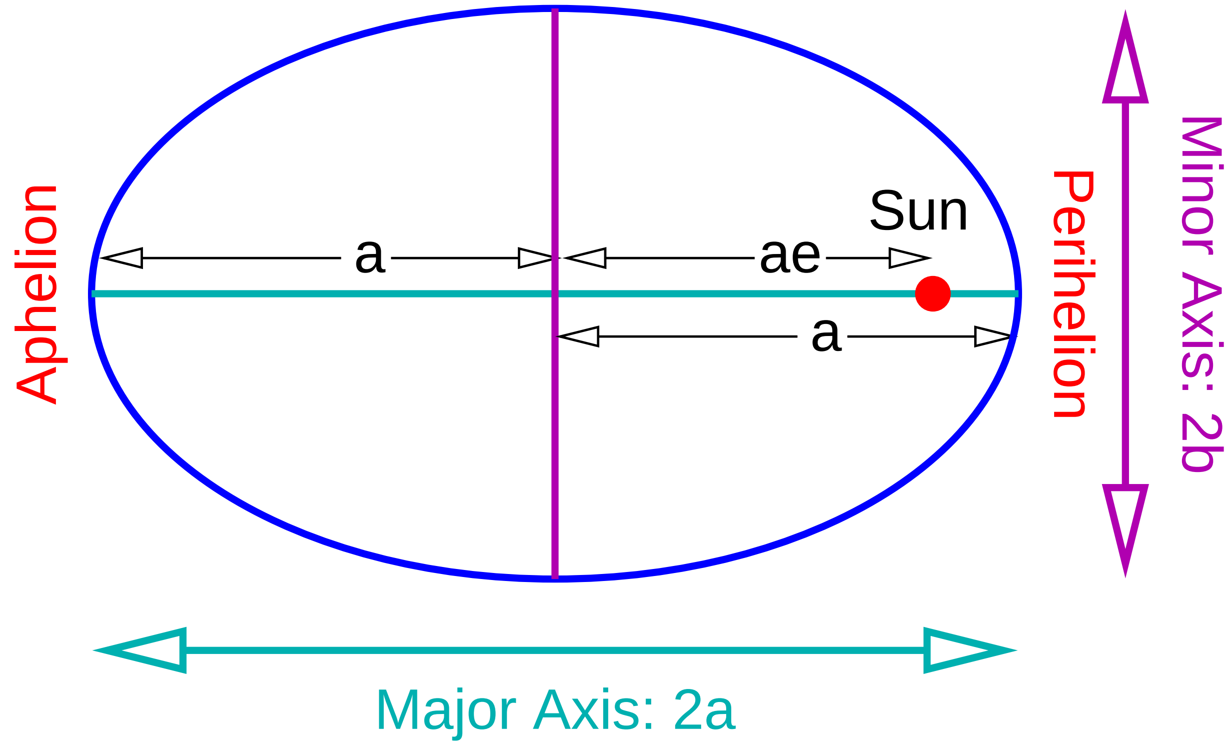
Law of cosines: $r'^2 = r^2 + (2ae)^2 - 2 \cdot r \cdot 2ae \cdot \cos(\pi - \theta)$

use $r + r' = 2a$ and solve for r to find the polar coordinate form of the ellipse:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{4.3}$$

Check this for yourself! θ is called the *true anomaly*.

Keplers 1st Law



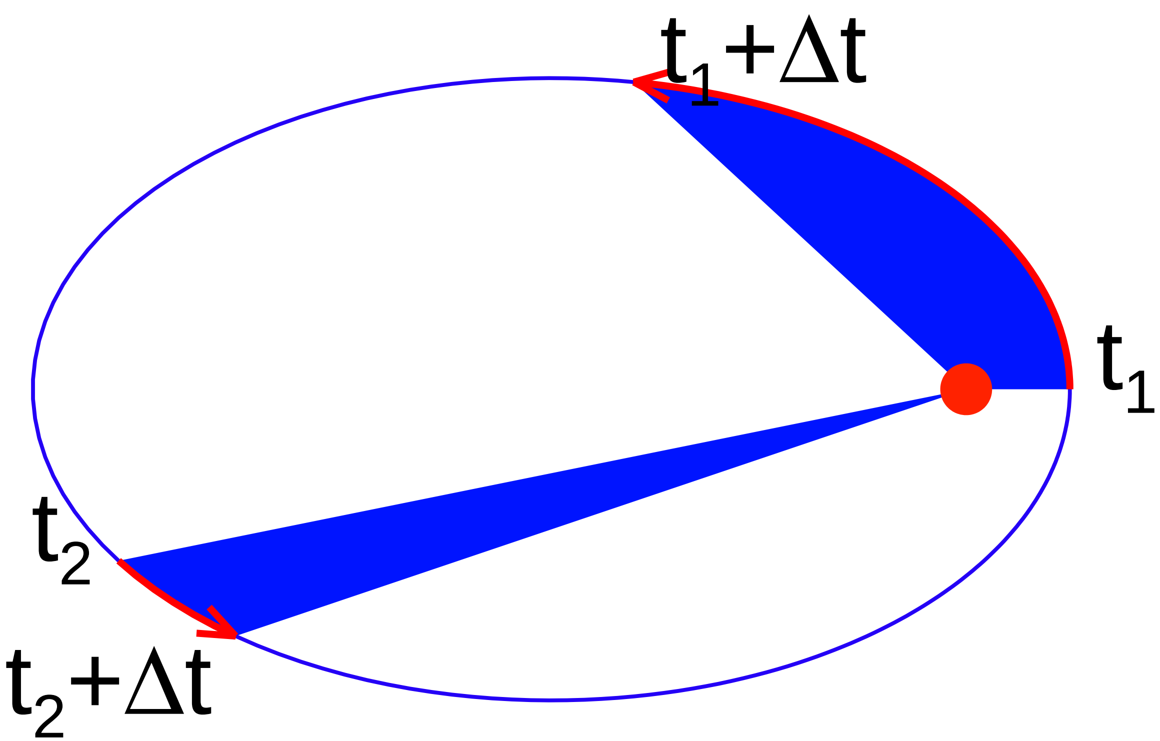
Finally, we need the closest and farthest point from a focus:

$$\begin{aligned}\text{closest point}: d_{\text{perihelion}} &= a - ae = a(1-e) \\ \text{farthest point}: d_{\text{aphelion}} &= a + ae = a(1+e)\end{aligned} \tag{4.4}$$

for stars: periastron and apastron,

for satellites circling the Earth: perigee and apogee.

2nd Law



Kepler's 2nd Law: The radius vector to a planet sweeps out equal areas in equal intervals of time.

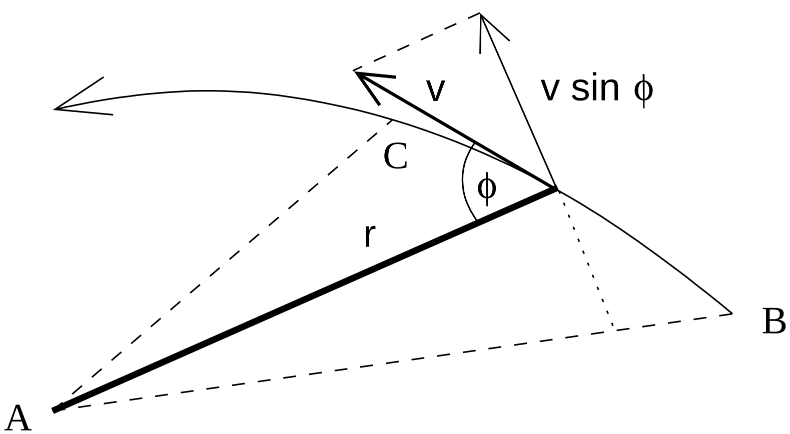
1. Kepler's 2nd Law is also called the *law of areas*.
2. perihelion: planet nearest to Sun $\Longrightarrow$ planet is fastest
3. aphelion: planet farthest from Sun $\Longrightarrow$ planet is slowest

Kepler's 2nd law is a direct consequence of the conservation of angular momentum. Remember that angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v} \tag{4.5}$$

and its absolute value is

$$L = mrv \sin\phi \tag{4.6}$$



To interpret the angular momentum, look at the figure at the left. The projection of the velocity vector perpendicular to the radius vector r is $v \sin\phi$. The distance traveled by the planet in an infinitesimally short time Δt is given by $\Delta x = \Delta t \cdot v \sin\phi$. Therefore, the area of the triangle ABC is given by

$$\Delta A = \frac{1}{2} r \Delta x = \frac{1}{2} r \Delta t v \sin\phi = \frac{L}{2m} \Delta t \tag{4.7}$$

Kepler's 2nd law states that the "sector velocity" $\mathrm{d}A/\mathrm{d}t$ is constant with time:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{L}{2m} = \text{const.} \tag{4.8}$$

To confirm that this claim is true, we need to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2m} \frac{\mathrm{d}L}{\mathrm{d}t} = 0 \tag{4.9}$$

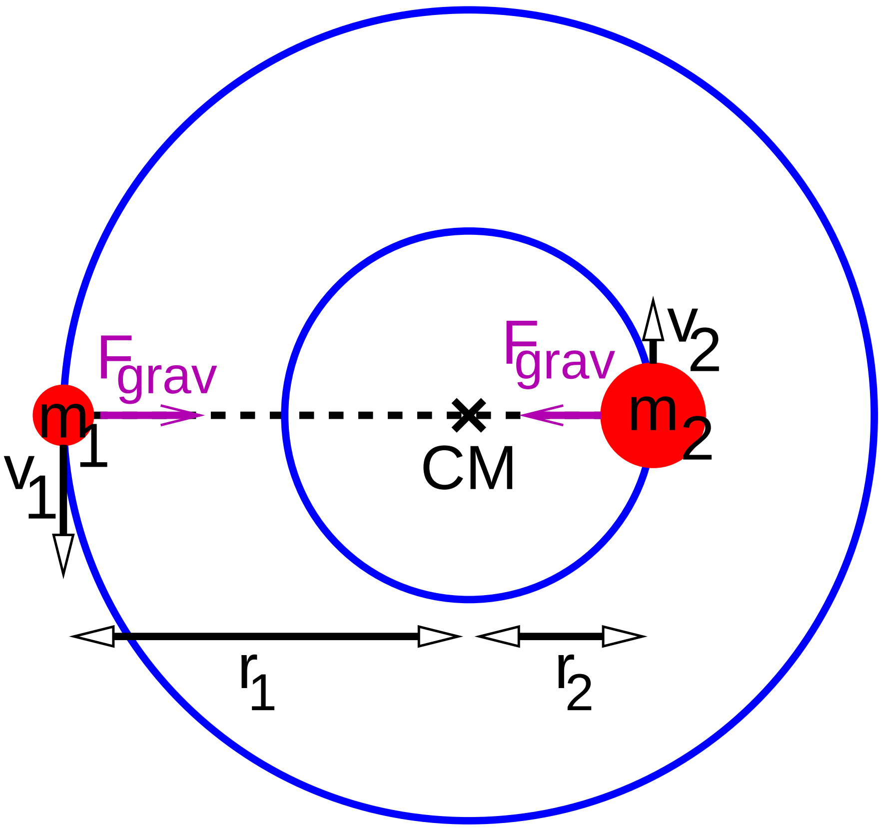
But $\mathrm{d}L/\mathrm{d}t$ is given by

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times \frac{GMm}{r^2} \frac{\boldsymbol{r}}{r} = 0 \tag{4.10}$$

since the cross product of a vector with itself is zero. Therefore, Kepler's 2nd law is true and is a consequence of the conservation of angular momentum for a central field.

3rd Law

Kepler's 3rd Law: The squares of the periods of the planets, P, are proportional to the cubes of the semimajor axes, a, of their orbits: $P^2 \propto a^3$.

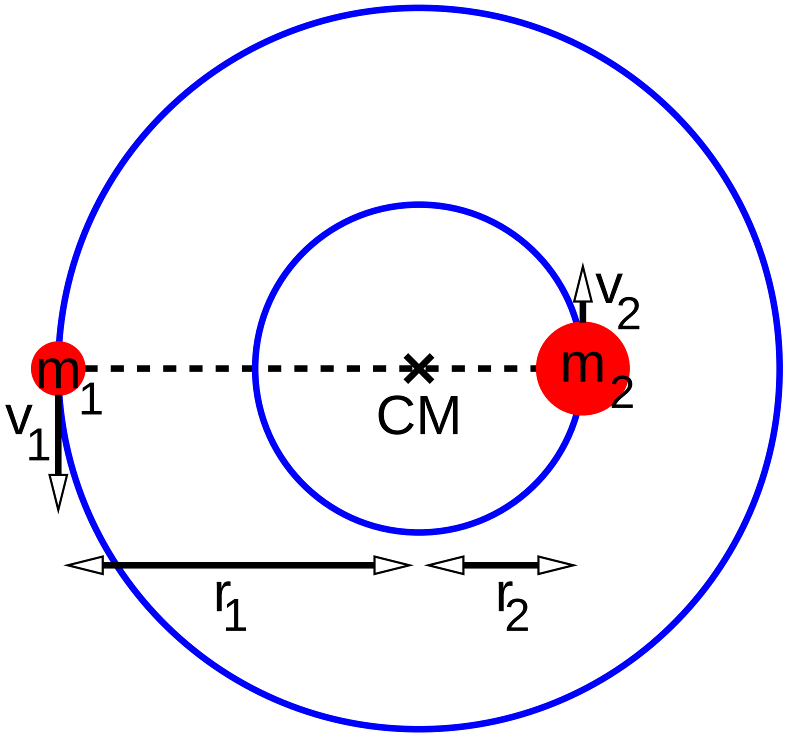


Calculating the motion of two bodies of mass m_1 and m_2 gives Newton's form of Kepler's third law:

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} R^3 \qquad (4.11)$$

where $r_1 + r_2 = R$ (for elliptical orbits: R is the semi-major axis).

Circular Motion



For an interpretation of Kepler's third law, consider the motion of two bodies with masses m_1 and m_2 on circular orbits with radii r_1 and r_2 around a point CM (see figure).

The reason for doing the computation with circular orbits is that the following discussion will be easier, however, all results from this section also apply to the general case of elliptical motion. This will be proven later in the lectures on Theoretical Mechanics.

The attractive force between the two points is given by Newton's law:

$$F_{\text{grav}} = G\frac{m_1 m_2}{R^2} = G\frac{m_1 m_2}{(r_1 + r_2)^2} \tag{4.12}$$

In order to keep the two bodies on circular orbits, the gravitational force needs to be equal the centripetal force keeping each body on its circular orbit.

The centripetal force is

$$\begin{aligned} F_{\text{cent},1} &= \frac{m_1 v_1^2}{r_1} = \frac{4\pi^2 m_1 r_1}{P^2} \\ F_{\text{cent},2} &= \frac{m_2 v_2^2}{r_2} = \frac{4\pi^2 m_2 r_2}{P^2} \end{aligned} \tag{4.13}$$

where $v = 2\pi r/P$ was used to determine the velocity of each of the bodies. Setting the centripetal force equal to the gravitational force gives

$$\begin{aligned} \frac{4\pi^2 m_1 r_1}{P^2} &= G\frac{m_1 m_2}{(r_1 + r_2)^2} \\ \frac{4\pi^2 m_2 r_2}{P^2} &= G\frac{m_1 m_2}{(r_1 + r_2)^2} \end{aligned} \tag{4.14}$$

canceling m_1 and m_2 results in

$$\begin{aligned} \frac{4\pi^2 r_1}{P^2} &= G\frac{m_2}{(r_1 + r_2)^2} \\ \frac{4\pi^2 r_2}{P^2} &= G\frac{m_1}{(r_1 + r_2)^2} \end{aligned} \tag{4.15}$$

Dividing these two equations by each other gives

$$\frac{r_1}{r_2} = \frac{m_2}{m_1} \quad \text{or} \quad m_1 r_1 = m_2 r_2 \tag{4.16}$$

This is the definition of the *center of mass*.

The total distance between the two bodies is

$$R = r_1 + r_2 = r_1 + \frac{m_1}{m_2} r_1 = r_1 \left(1 + \frac{m_1}{m_2}\right) \tag{4.17}$$

Inserting into one of the equations 4.15 gives

$$\frac{4\pi^2}{P^2} \cdot R \cdot \frac{m_2}{m_1 + m_2} = \frac{G m_2}{R^2} \tag{4.18}$$

such that

$$\frac{4\pi^2}{P^2} = \frac{G(m_1 + m_2)}{R^3} \quad \text{or} \quad P^2 = \frac{4\pi^2}{G(m_1 + m_2)} R^3 \tag{4.19}$$

This is *Newton's form of Kepler's 3rd law*.

3rd Law

Newton's form of Kepler's 3rd law is the most general form of the law.

However, often shortcuts are possible.

Assume one central body dominates, $m_1 = M \gg m_2$:

$$\frac{P^2}{a^3} = \frac{4\pi^2}{GM} = \text{const.} = k \tag{4.20}$$

So, if we know P and a for one body moving around m_1, can calculate k.

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For the Solar System, use Earth:

- $P_\oplus = 1\,\text{year}$ (by definition!)
- $a_\oplus = 1\,\text{AU}$ (Astronomical Unit, $1\,\text{AU} = 149.6 \times 10^6\,\text{km}$)

$\Longrightarrow k = 1\,\text{yr}^2\,\text{AU}^{-3}$

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Answer: $P_{♃}^2 = 1\,\text{yr}^2\,\text{AU}^{-3} \cdot 5.2^3\,\text{AU}^3 \sim 140\,\text{yr}^2$, or $P_{♃} \sim 12\,\text{years}$

(with pocket calculator: $P_{♃} = 11.86\,\text{years}$)

N-Body Problem, I

In reality, planets also excert forces onto each other.

Total equation of motion for the i-th object:

$$m_i \ddot{\boldsymbol{r}}_i = -\sum_{k=1}^{N} \frac{G m_i m_k}{r_{i,k}^2} \frac{\boldsymbol{r}_i - \boldsymbol{r}_k}{r_{i,k}} \tag{4.24}$$

$\Longrightarrow$ 3N differential equations of 2nd order, requiring 6N integrations for their solution.

Closed solution only possible for 10 of these (6: from motion of center of mass, 3: conservation of angular momentum, 1: conservation of energy).

Analytic solution: "Perturbation theory":

1. Assume two body motion around Sun for all planets
2. Evaluate force based on this motion.
3. Update positions with this "perturbation".
4. Iterate (i.e., goto step 2)

N-Body Problem, II

Perturbation theory yields two kinds of perturbations:

periodic perturbations: Terms containing time in $\sin$- and $\cos$-functions.

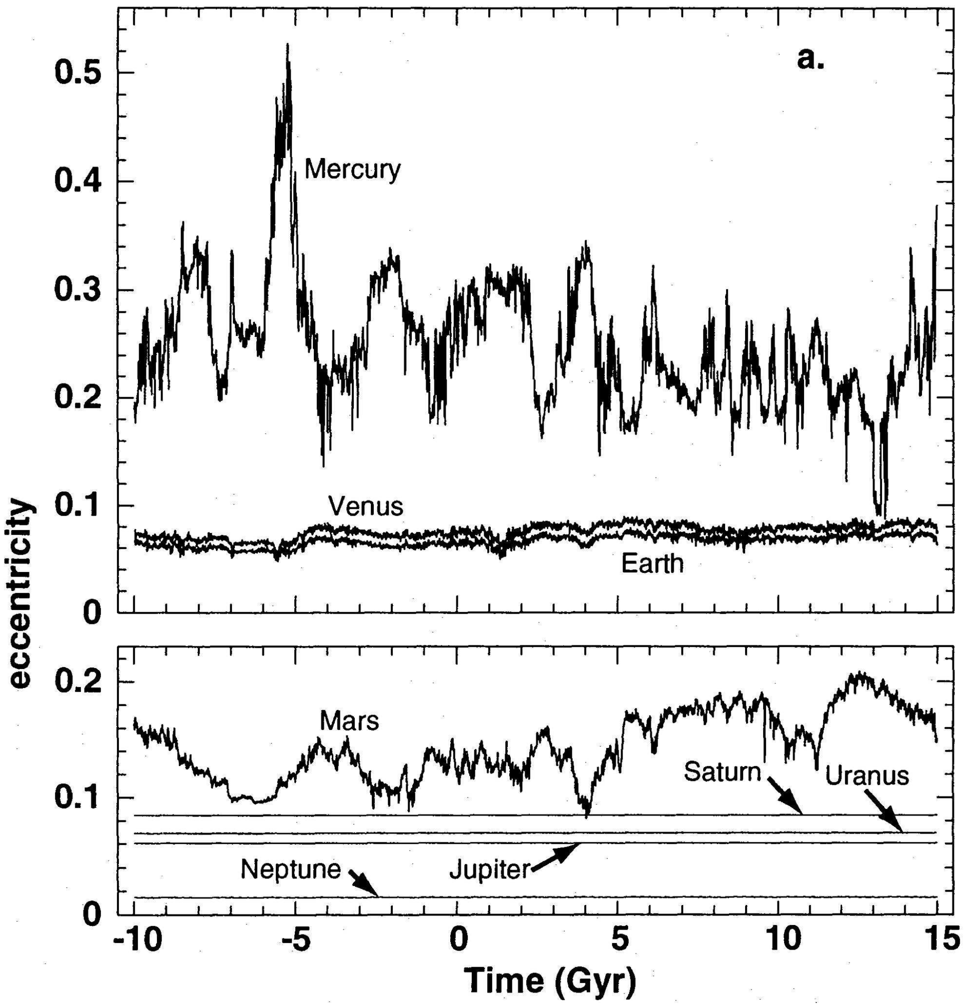
secular perturbations: Long term changes which depend on time (usually as a polynomial).

Analytical approach is very important for understanding the underlying physics, but mathematically *very* tedious. Series do not converge on long timescales (1000's of years).

$\Longrightarrow$ New high precision calculations are all based on numerical simulations, i.e., direct solution of equation of motion on computers.

Today's standard: DE102, DE405, DE414 from Jet Propulsion Laboratory, Pasadena, and INPOP06 from Laskar et al., IMCCE, Observatoire de Paris.

Long-Term Evolution of the Solar System



(Laskar, 1994)

Numerical simulations allow to obtain good information about behavior of solar system for timescales of a few 10 Million years around the present $\Longrightarrow$ Important, e.g., for paleoclimatology.

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"Chaotic": Initial errors get amplifi ed exponentially, here by factor of 10 on time scales of $\sim$10 million years.

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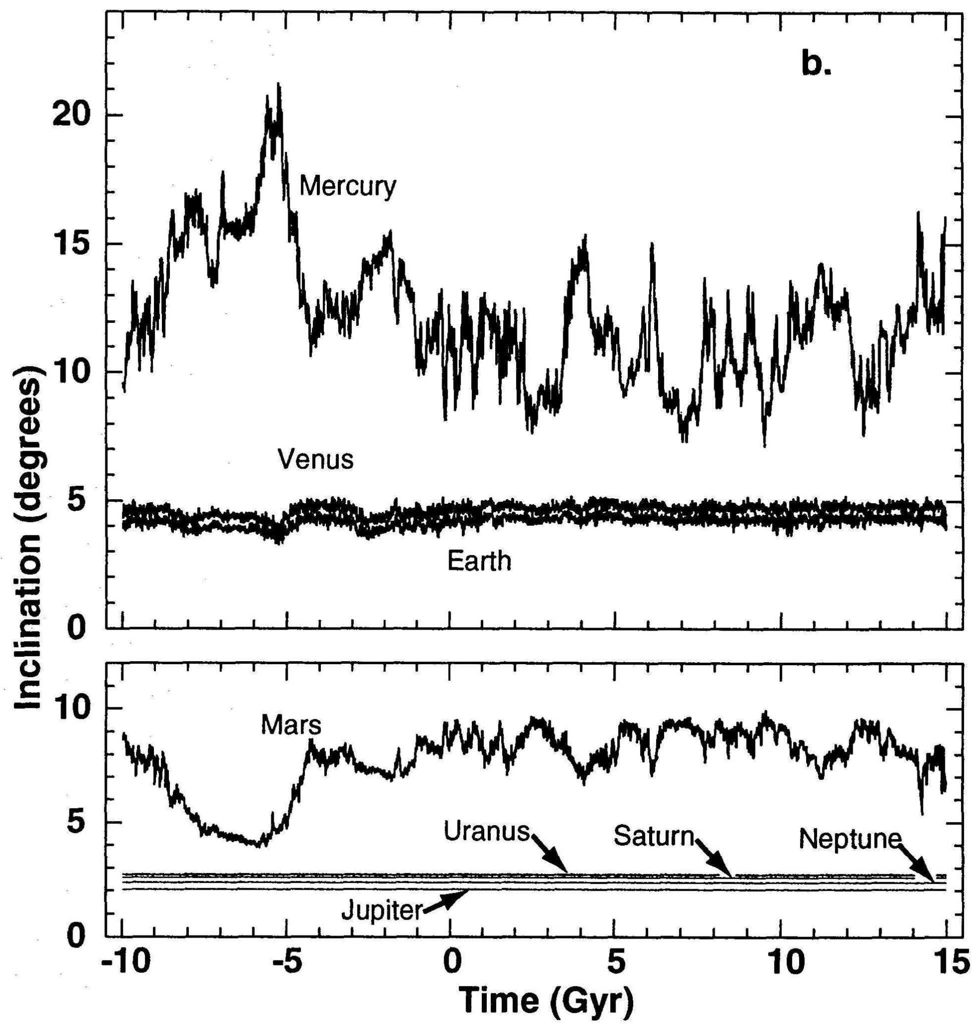

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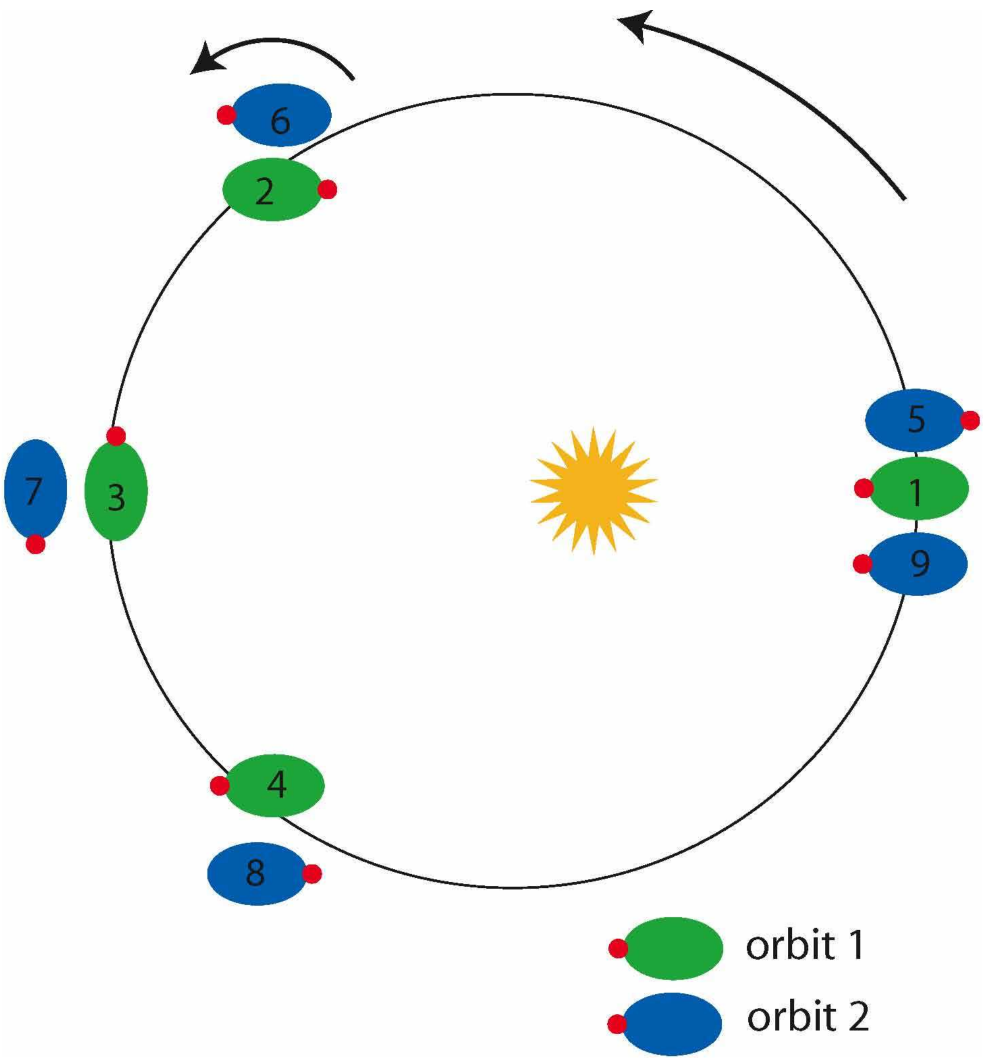

(courtesy J. Laskar/CNRS)

Rotation period and orbital period of Mercury are in a 3:2 resonance.

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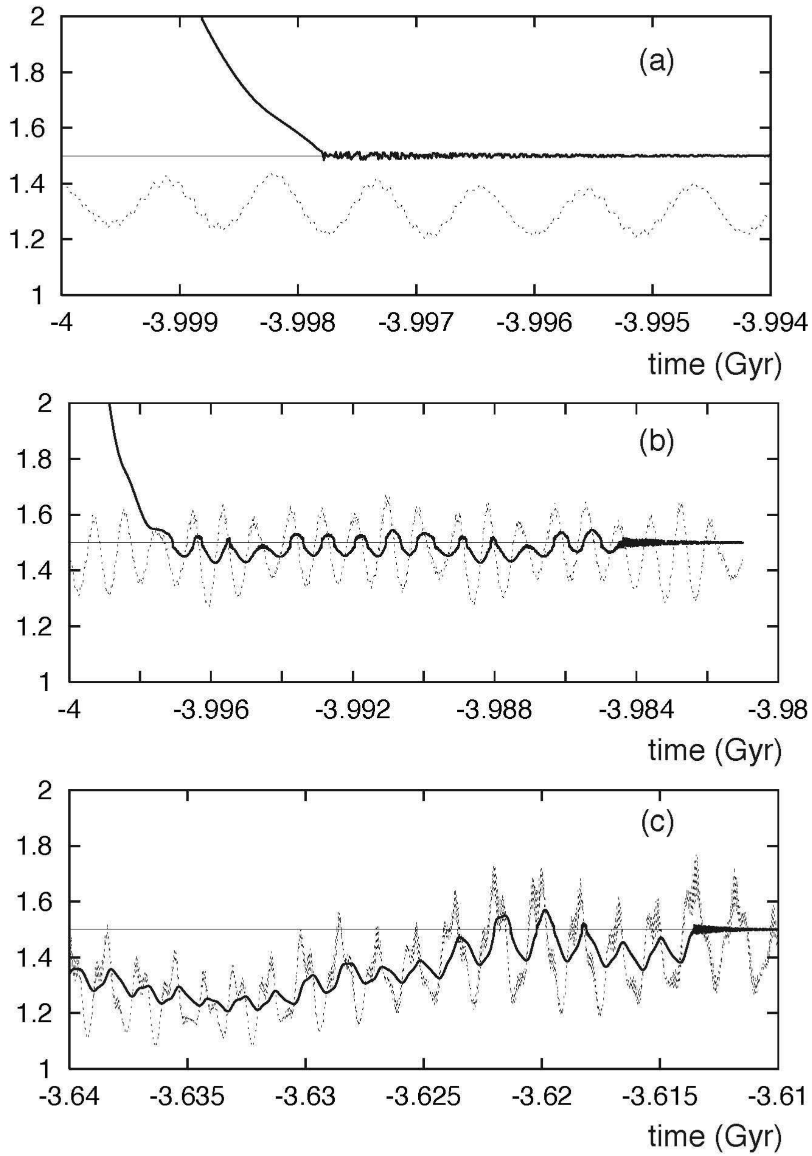

Chaotic motion of Mercury's orbit increases probability of capturing Mercury in its 3:2 resonance with its orbit via tidal dissipation from $<$5% in classical theory to $\sim$55%.

Similar explanation also for retrograde rotation of Venus, Earth is saved from such a behavior because of the stabilizing influence of the Moon.

(Correia & Laskar, 2004)

Correia, A. C. M., & Laskar, J., 2004, Nature, 429, 848

Laskar, J., 1989, Nature, 338, 237

Laskar, J., 1990, Icarus, 88, 266

Laskar, J., 1994, A&A, 287, L9

Planets: Atmospheres