Blackbody Radiation

Introduction

First radiation process to look at: radiation in thermal equilibrium with itself: blackbody radiation

Most important radiation process of the universe, as it is responsible for the spectral shape of stellar spectra.

There are two ways to derive the spectrum of a black body:

1. classical way: Calculate number of photons in phase space in thermodynamic equilibrium
   This is the original derivation of the BB spectrum, first performed by Max Planck.
2. quantum mechanics: Calculate properties of radiation field assuming it is quantized.
   This is a more elegant and less technical derivation due to Albert Einstein.

We will use the 2nd way here (following Padmanabhan).

Quantization, I

Quantum mechanics: (most) things are quantized, i.e., have discrete states.

Look at an atom absorbing a photon:

- Before absorption: atom in ground state with n photons: $|G\rangle|n\rangle$.
- After absorption: atom in excited state with $n-1$ photons: $|E\rangle|n-1\rangle$.

If the probability for absorption of one photon is denoted with Q, then the probability for a transition between the states is

$$\mathcal{P}\left(|G\rangle|n\rangle \to |E\rangle|n-1\rangle\right) \propto Qn \tag{3.1}$$

For the emission of a photon:

$$\mathcal{P}\left(|E\rangle|m\rangle \to |G\rangle|m+1\rangle\right) \propto Qn = Q(m+1) \tag{3.2}$$

since the probability going into both directions is the same (microreversibility).

Quantization, II

The transition rates between the two levels are:

$$R_{\text{abs}} = Qn \tag{3.3}$$

$$R_{\text{em}} = Q(n+1) = Qn + Q \tag{3.4}$$

One calls:

- Q: *coefficient of* spontaneous *emission*
- Qn: *coefficient of* stimulated *emission*

Due to tradition, we do not use photon numbers or photon densities such as n, but specific intensity et al.

$\Longrightarrow$ need to translate from QM picture to classical picture

Photons and I, I

The number of photons in a phase space cell $(\boldsymbol{p}, \boldsymbol{p} + \mathrm{d}\boldsymbol{p})$ is:

$$\mathrm{d}N = 2nV \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \tag{3.5}$$

Note: now n is a density, but this does not matter since we'd just have to divide all equations before this slide by an (arbitrary) volume.

Energy flowing through volume $\mathrm{d}^3x = A(c\,\mathrm{d}t)$:

$$\mathrm{d}E = h\nu\,\mathrm{d}N = 2nh\nu\,\mathrm{d}A(c\,\mathrm{d}t)\frac{d^3p}{h} \tag{3.6}$$

Now note that for photons

$$|p| = \frac{h\nu}{c} \quad \text{and} \quad \mathrm{d}^3p = p^2\,\mathrm{d}p\,\mathrm{d}\Omega = \left(\frac{h}{c}\right)^3 \nu^2\,\mathrm{d}\nu\,\mathrm{d}\Omega \tag{3.7}$$

such that

$$\mathrm{d}E = \frac{2h\nu^3}{c^2}n\;\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \tag{3.8}$$

Photons and I, II

We just found

$$\mathrm{d}E = \frac{2h\nu^3}{c^2}n\;\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \tag{3.8}$$

but compare this to the definition of specific intensity:

$$\mathrm{d}E = I_\nu\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \tag{2.48}$$

and therefore

$$I_\nu = \frac{2h\nu^3}{c^2}n \tag{3.9}$$

Two remarks:

1. Since $I_\nu/\nu^3 \propto n$, I_ν/ν^3 is Lorentz invariant
   (since photon number is preserved in all frames of reference)
2. The above is (yet) another proof that I_ν is independent of distance
   (since photon number does not change)

Einstein coefficients

Interlude: The definitions we had so far can be used to introduce the so-called Einstein coefficients A and B, which are often used, e.g., in laser theory and in atomic physics.

$$R_{\mathrm{abs}} =: BI_\nu = Qn \tag{3.10}$$
$$R_{\mathrm{em}} =: A + BI_\nu = Q(n+1) \tag{3.11}$$

Since

$$I_\nu = \frac{2h\nu^3}{c^2}n \tag{3.9}$$

we have

$$Q = \frac{2h\nu^3}{c}B \quad \text{and} \quad \frac{A}{B} = \frac{2h\nu^3}{c^2} \tag{3.12}$$

$\Longrightarrow$ If *one* of Q, A, and B is known, can calculate the other quantities from it.

We will take a look at how this is done (much) later in this course.

Note that the derivation shown here makes the process of "stimulated emission" far less mystical than it often appears (at least to me...).

Derivation of BB spectrum

We now have everything collected to derive the spectrum of radiation in thermal equilibrium, i.e., the spectrum of black body radiation.

Radiative equilibrium implies

$$\text{Absorption rate} = \text{Emission rate} \tag{3.13}$$
$$N_{\mathrm{G}}Qn = N_{\mathrm{E}}Q(n+1) \tag{3.14}$$

where N_{E} and N_{G} number of atoms in ground and excited state. Solving for n:

$$n = \frac{1}{(N_{\mathrm{G}}/N_{\mathrm{E}}) - 1} \tag{3.15}$$

But thermodynamic equilibrium implies Boltzmann statistics, i.e.,

$$\frac{N_{\mathrm{E}}}{N_{\mathrm{G}}} = \exp\left(-\frac{\Delta E}{kT}\right) \tag{3.16}$$

such that

$$n = \frac{1}{\exp(\Delta E/kT) - 1} \tag{3.17}$$

Derivation of BB spectrum

The photon number density was

$$n = \frac{1}{\exp(\Delta E/kT) - 1} \tag{3.17}$$

We could have “known” this result since this is just the occupation number of bosons with a chemical potential $\mu = 0$.

Because

$$I_\nu = \frac{2h\nu^3}{c^2} n \tag{3.9}$$

this means that for BB-radiation

$$\frac{\mathrm{d}E}{\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu} = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1} \tag{3.18}$$

(since $\Delta E = h\nu$). This is the so-called Planck spectrum.

The classical derivation of the Planck spectrum is much less straightforward than the quantum-mechanical approach, however, it is instructive also to look at this approach to derive Eq. (3.18) as it shows the whole origin of the spectrum from a completely different angle.

The basic assumptions entering the calculation are

1. Photons are Bosons, i.e., more than one photon per phase space cell possible.
2. Photons are in thermodynamic equilibrium at all frequencies.

In order to derive the BB-spectrum from these assumptions we will perform the following steps:

1. Calculate the mean energy of photons of frequency ν in phase space cell, $\langle E(\nu)\rangle$
2. Calculate the number of phase space cells as a function of frequency, $N(\nu)$.
3. Calculate the photon spectrum as the product $\langle E(\nu)\rangle \cdot N(\nu)$.

First step: Calculation of the mean energy of photons of frequency ν in on phase space cell.

We describe the phase space cell as a box, and therefore we can describe the photons as quantum-mechanical particles in a box. Because of the properties of the quantum-mechanical harmonic oscillator, the total energy energy of box with n photons is given by

$$E_n = \left(n + \frac{1}{2}\right) \cdot h\nu \tag{3.19}$$

where $\frac{1}{2}h\nu$ is the ground state energy, which is unobservable.

The probability that the oscillator is in nth state is given again by the Boltzmann distribution

$$P_n(\nu, T) = \frac{\exp\left(-\left(n+\frac{1}{2}\right)h\nu\right)}{\sum_{n'}\exp\left(-\left(n'+\frac{1}{2}\right)h\nu\right)} = \frac{\exp(-nh\nu/kT)}{\sum_{n'}\exp(-n'h\nu/kT)} \tag{3.20}$$

Therefore, the average energy per phase cell is

$$\langle E\rangle = \sum_n E_n P_n(\nu, T) \tag{3.21}$$

$$= \sum_n \left\{\left(n + \frac{1}{2}\right) h\nu \cdot \frac{\exp(-nh\nu/kT)}{\sum_{n'}\exp(-n'h\nu/kT)}\right\} \tag{3.22}$$

introducing $x = h\nu/kT$

$$= \frac{kT\sum_n \left(n+\frac{1}{2}\right) x\exp(-nx)}{\sum_n \exp(-nx)} \tag{3.23}$$

$$= kT\left\{\frac{\sum_n nx\exp(-nx)}{\sum_n \exp(-nx)} + \frac{x}{2}\right\} \tag{3.24}$$

To evaluate $\langle E\rangle$ we need to compute the geometric sums $\sum_n \exp(-nx)$ and $\sum_n nx\exp(-nx)$. To do so, look at the Taylor series of $f(y) = (1-y)^{-1}$: By induction we find

$$f(y) = (1-y)^{-1} \tag{3.25}$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} = \frac{(-1)(-1)}{(1-y)^2} = \frac{1}{(1-y)^2} \tag{3.26}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} = \frac{(-1)(-2)}{(1-y)^3} = \frac{1\cdot 2}{(1-y)^3} \tag{3.27}$$

and in general

$$\frac{\mathrm{d}^n f}{\mathrm{d}y^n} = \frac{n!}{(1-y)^{n+1}} \tag{3.28}$$

Therefore, the Taylor series of $f(y)$ around $y = 0$ is

$$\frac{1}{1-y} = \sum_n \frac{1}{n!}\left.\frac{\mathrm{d}^n f}{\mathrm{d}y^n}\right|_{y=0} y^n = \sum_n y^n \tag{3.29}$$

Substituting $y = \exp(-x)$ then shows that

$$\sum_n \exp(-nx) = \frac{1}{1-\exp(-x)} \tag{3.30}$$

To evaluate the second sum, take a look at

$$\sum_n nx\exp(-nx) = x\sum_n n\exp(-nx) \tag{3.31}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_n \exp(-nx) = -\sum_n n\exp(-nx) \tag{3.32}$$

such that

$$\sum_n n\exp(-nx) = -\frac{\mathrm{d}}{\mathrm{d}x}\sum_n \exp(-nx) \tag{3.33}$$

by Eq. (3.30)

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-\exp(-x)}\right) \tag{3.34}$$

$$= \frac{\exp(-x)}{(1-\exp(-x))^2} \tag{3.35}$$

Multiplying with x then shows that

$$\sum_n nx\exp(-nx) = \frac{x\exp(-x)}{(1-\exp(-x))^2} \tag{3.36}$$

Because of Eq. 3.30 and 3.36,

$$\langle E\rangle = kT\left(\frac{x\,\mathrm{e}^{-x}\left(1-\mathrm{e}^{-x}\right)^{-2}}{\left(1-\mathrm{e}^{-x}\right)^{-1}} + \frac{x}{2}\right) \tag{3.37}$$

$$= \frac{h\nu\exp(-x)}{1-\exp(-x)} + \frac{h\nu}{2} \tag{3.38}$$

$$= \frac{h\nu}{\mathrm{e}^{h\nu/kT}-1} + \frac{h\nu}{2} \tag{3.39}$$

And note again that the $h\nu/2$ term is unobservable, i.e., the zero point of energy can be set to get rid of this term.

$$n_\gamma(\nu, T) = \frac{\langle E\rangle}{h\nu} = \frac{1}{\exp(h\nu/kT) - 1} \tag{3.17}$$

Second Step: Computation of density of phase space cells in box L_x, L_y, L_z.

The wave vector of a photon is

$$\boldsymbol{k} = \frac{2\pi}{\lambda}\boldsymbol{n} = \frac{2\pi\nu}{c}\boldsymbol{n} \tag{3.40}$$

To get all possible photons we need to count the number of al distinguishable photons at the same frequency, i.e., photons with different spin or with a different number of nodes (=different n).

Spin is the easy one: there are only 2 polarization states.

To calculate the number of nodes, look the number in the x-, y-, or z-direction. In either direction

$$n_x = \frac{L_x}{\lambda} = \frac{k_x L_x}{2\pi} \iff \mathrm{d}n_x = \frac{L_x}{\lambda} = \frac{L_x\,\mathrm{d}k_x}{2\pi} \qquad (3.41)$$

For $n \gg 1$, we can go to a "continuum of states". Eq. 3.41 then implies

$$\mathrm{d}N = \mathrm{d}n_x\,\mathrm{d}n_y\,\mathrm{d}n_z = \frac{L_x L_y L_z\,\mathrm{d}^3k}{(2\pi)^3} = \frac{V\,\mathrm{d}^3k}{(2\pi)^3} \qquad (3.42)$$

Therefore, the total number of states per unit volume and per wave number is

$$\frac{n_k}{\mathrm{d}^3k} = 2\cdot\frac{\mathrm{d}N}{V}\frac{1}{\mathrm{d}^3k} = \frac{2}{(2\pi)^3} \qquad (3.43)$$

where the factor 2 is due to spin.

Because of Eq. (3.40),

$$\mathrm{d}^3k = k^2\,\mathrm{d}k\,\mathrm{d}\Omega = \frac{(2\pi)^3}{c^3}\nu^2\,\mathrm{d}\nu\,\mathrm{d}\Omega \qquad (3.44)$$

such that the density of states, i.e., the number of states per solid angle, volume, and frequency is given by

$$\rho_\mathrm{s} = \frac{n_\nu}{\mathrm{d}\nu\,\mathrm{d}\Omega} = \frac{2}{(2\pi)^3}\cdot\frac{(2\pi)^3}{c^3}\nu^2 = \frac{2\nu^2}{c^3} \qquad (3.45)$$

Third step: Black Body Spectrum

To summarize, we had

- the mean energy of the state:

$$\langle E\rangle = \frac{h\nu}{e^{h\nu/kT}-1} \qquad (3.39)$$

- the state density:

$$\rho_\mathrm{s} = \frac{2\nu^2}{c^3} \qquad (3.45)$$

The total energy density is then

$$u_\nu(\Omega) = \langle E\rangle\cdot\rho_\mathrm{s} = \frac{2h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT)-1} \qquad (3.46)$$

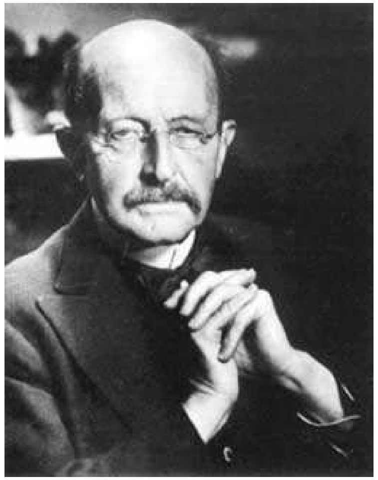
(energy per volume per frequency per solid angle)

Because of Eq. (2.71) ($u_\nu = I_\nu/c$), the intensity is given by

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1} =: B_\nu \qquad (3.18)$$

This is the spectrum of a black body.

Spectrum, I



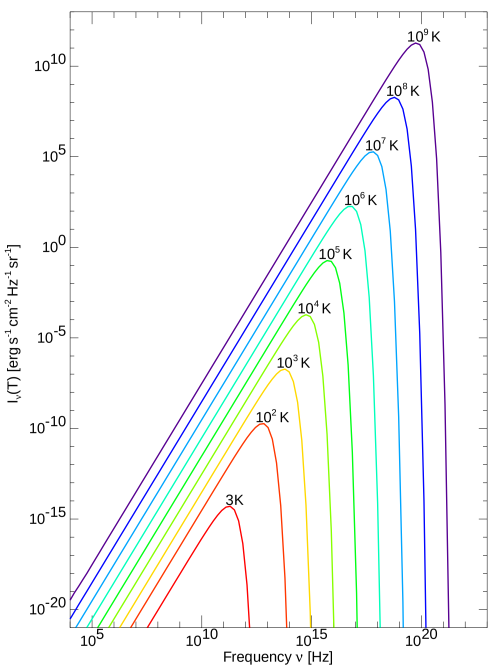
Max Planck (1858–1947)

In wavelength space, the spectrum of a black body is blackbody radiation:

$$\frac{\mathrm{d}E}{\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\lambda} = B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT)-1} \qquad (3.47)$$

B_λ: Energy emitted per second and *wavelength* interval

- $h = 6.63\times10^{-27}$ erg s: Planck's constant
- $k = 1.38\times10^{-16}$ erg K^{-1}: Boltzmann constant



Rayleigh-Jeans Law

John William Strutt, Lord Rayleigh (1842–1919); Nobel prize 1904 for the discovery of argon

For $h\nu \ll kT$ ($\nu \lesssim 2\times 10^{10}\,\mathrm{Hz}\cdot(T/1\,\mathrm{K})$),

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + \ldots \tag{3.48}$$

such that

$$B_\nu \sim \frac{2\nu^2}{c^2}kT \quad\Longleftrightarrow\quad B_\lambda \sim \frac{2c}{\lambda^4}kT \tag{3.49}$$

This is the Rayleigh-Jeans law.

See worksheet 2.

Radio astronomers use the Rayleigh-Jeans law to define the brightness temperature,

$$T_\mathrm{b} = I_\nu \cdot \frac{c^2}{2k\nu^2} \tag{3.50}$$

as a measure of radio intensity I_ν.

The λ^4 proportionality was found empirically by Lord Rayleigh in 1900 and published by Rayleigh and James Jeans in 1905 (*after* Planck!), but suffered from the “Ultraviolet catastrophe”.

Wien Spectrum

Wilhelm Wien (1864–1928)
Nobel prize 1911

For $h\nu \gg kT$, ($\nu \gtrsim 2\times 10^{10}\,\mathrm{Hz}\cdot(T/1\,\mathrm{K})$),

$$\exp\left(\frac{h\nu}{kT}\right) - 1 \sim \exp\left(\frac{h\nu}{kT}\right) \tag{3.51}$$

such that

$$B_\nu \sim \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right) \tag{3.52}$$

the Wien spectrum (or Wien’s law).

Wien’s law is the result one obtains for the spectrum of black body radiation assuming classical thermodynamics, it was historically found by Wilhelm Wien in 1896, before the publication of Planck’s results, and was proven to work well in the UV, but failed in the IR.

Wien Displacement Law

The frequency of maximum intensity, ν_max is obtained by solving

$$\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu=\nu_\mathrm{max}} = 0 \tag{3.53}$$

which is equivalent to solving

$$x = 3(1 - \exp(-x)) \tag{3.54}$$

where $x = h\nu_\mathrm{max}/kT$. Numerically, $x = 2.82$, therefore

$$h\nu_\mathrm{max} = 2.82 \cdot kT \tag{3.55}$$

This is the Wien displacement law.

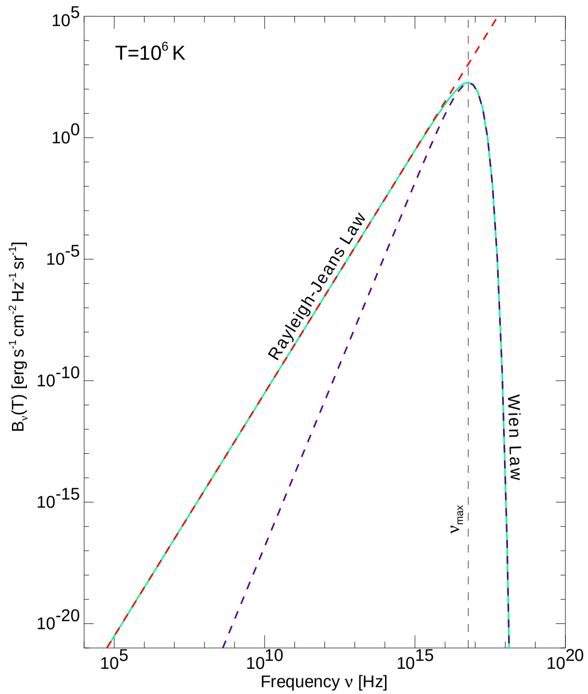
The frequency of maximum flux is directly proportional to the black body temperature.

Likewise, for B_λ, one finds

$$\lambda_\mathrm{max}T = 0.2898\,\mathrm{cm\,K} \tag{3.56}$$

Note that $\lambda_\mathrm{max}\nu_\mathrm{max} \neq c$!

Do not confuse Wien’s law and the Wien displacement law...



- Rayleigh-Jeans applies for $\nu \lesssim \nu_\mathrm{max}$
- Wien applies for $\nu \gtrsim \nu_\mathrm{max}$.

Stefan-Boltzmann law

Ludwig Boltzmann
(1844–1906)

Jožef Stefan
(1835–1893)

As shown on worksheet 2, the total energy density of black body radiation is given by the Stefan-Boltzmann law

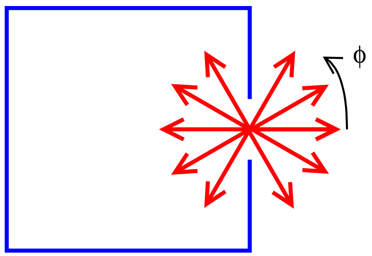
$$u_{\mathrm{BB}}(T) = \int_0^\infty \frac{4\pi}{c} B_\nu \,\mathrm{d}\nu = \frac{8\pi^5}{15}\left(\frac{kT}{hc}\right)^3 kT =: aT^4 = \left(\frac{T}{3400\,\mathrm{K}}\right)^4 \mathrm{erg\,cm^{-3}} \quad (3.57)$$

where the radiation density constant,

$$a := \frac{8\pi^5 k^4}{15c^3h^3} = 7.566 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}} \quad (3.58)$$

Note: units of $u_{\mathrm{BB}}(T)$ are erg cm^{-3}.

Stefan-Boltzmann law



Often we are interested in the radiation diffusing out of a medium in thermal equilibrium, such as the flux of radiation at the surface of a star.

$$F = \int_0^{2\pi}\int_0^{\pi/2}\int_0^\infty B_\nu \,\mathrm{d}\nu\,\mathrm{d}\phi\,\mathrm{d}\theta \quad (3.59)$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{cu_{\mathrm{BB}}}{4\pi}\cos\phi\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \quad (3.60)$$

$$= \frac{cu_{\mathrm{BB}}}{2}\int_0^{\pi/2}\frac{1}{2}\cos(2\phi)\,\mathrm{d}\phi = \frac{cu_{\mathrm{BB}}}{4} \quad (3.61)$$

where $\cos\phi$ is the projection factor between the direction of B_ν and the area (Lambert's law).

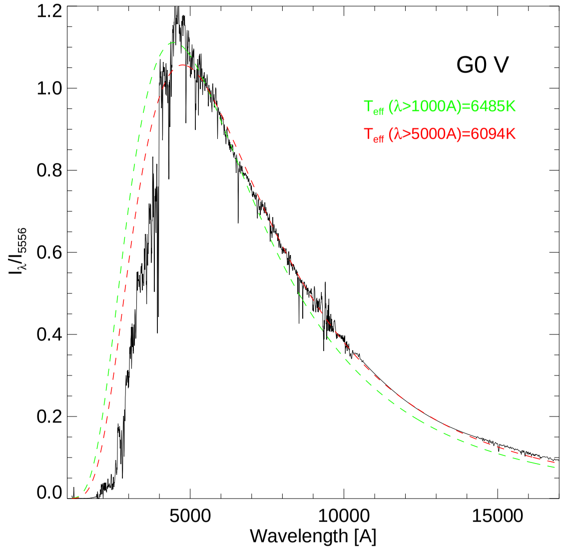
The total flux emitted by the surface is therefore given by

$$F = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} = \frac{c}{4}u_{\mathrm{BB}} = \frac{ac}{4}T^4 =: \sigma_{\mathrm{SB}}T^4 \quad (3.62)$$

with the Stefan-Boltzmann constant

$$\sigma_{\mathrm{SB}} := \frac{2\pi^5k^4}{15c^2h^3} = 5.671 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,K^{-4}\,s} \quad (3.63)$$

Effective Temperature



The effective temperature, T_{eff}, of an object with spectral shape I_ν is the temperature for which

$$F = \int I_\nu \cos\theta\,\mathrm{d}\nu\,\mathrm{d}\Omega = \sigma T_{\mathrm{eff}}^4 \quad (3.64)$$

Sometimes, I_ν is only known over a certain wavelength range, and depending on the spectrum the *measured* T_{eff} will depend on this range (see figure).

G0 v spectrum after Pickles (1998), PASP 110, 863

Planetary Surface Temperatures

The temperature of an irradiated body is given from energy equilibrium:

$$\frac{L_\odot}{4\pi a^2}\pi r^2 = \sigma_{\mathrm{SB}}T^4 4\pi r^2 \quad (3.65)$$

where a: distance to Sun, r: planetary radius.

Therefore

$$T = \left(\frac{L_\odot}{16\pi\sigma_{\mathrm{SB}}r^2}\right)^{1/4} = \frac{281\,\mathrm{K}}{(a/1\,\mathrm{AU})^{1/2}} \quad (3.66)$$

Last step used $L_\odot = 4 \times 10^{33}$ erg s^{-1} and 1 AU $= 1.496 \times 10^{13}$ cm.

If the planet reflects part of the radiation and if the IR emissivity is only roughly a BB, then Eq. (3.65) is modified,

$$(1-B)\frac{L_\odot}{4\pi a^2}\pi r^2 = \epsilon\sigma_{\mathrm{SB}}T^4 4\pi r^2 \implies T = \frac{281\,\mathrm{K}}{(a/1\,\mathrm{AU})^{1/2}}\left(\frac{1-B}{\epsilon}\right)^{1/4} \quad (3.67)$$

where B: Bond albedo, and ϵ: effective emissivity

For the Earth, $B = 0.39$, for Venus, $B = 0.72$. Thus, since $T_{\mathrm{Earth}} \sim 288$ K, $\epsilon_{\mathrm{Earth}} = 0.55 < 1$ (greenhouse effect).

If the planet is not a fast rotator, replace $4\pi r^2$ by $2\pi r^2$.