Synchrotron Radiation

Introduction

Synchrotron-Radiation (=Magnetobremsstrahlung): Radiation emitted by relativistic electrons in a magnetic field.

For non-relativistic particles, this process is called cyclotron radiation.

Goal: Qualitative analysis of the spectral features as detailed computations quite complicated.

Literature:

Ginzburg & Syrovatskii 1965, ARAA 3, 297
Blumenthal & Gould 1970, RMP 42, 237
Reynolds 1982, ApJ 256, 13

Motion

Lorentz-Force ($\boldsymbol{E} = 0$)

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{e}{c}\boldsymbol{v} \times \boldsymbol{B} \quad \text{where} \quad \boldsymbol{p} = \frac{m_\mathrm{e}\boldsymbol{v}}{\sqrt{1-\beta^2}} = \gamma m_\mathrm{e}\boldsymbol{v} \tag{6.1}$$

where $\beta = v/c$.

Assumption: No radiative losses (i.e., electron does *not* emit synchrotron radiation...): $\gamma = \text{const.}$

Velocity-vector of the electron:

$$\boldsymbol{v}_\parallel = \frac{\boldsymbol{v}\cdot\boldsymbol{B}}{B}\frac{\boldsymbol{B}}{B} \qquad \boldsymbol{v}_\perp = \frac{\boldsymbol{B}\times(\boldsymbol{v}\times\boldsymbol{B})}{B^2} \tag{6.2}$$

$$|\boldsymbol{v}_\parallel| = v\cos\alpha \qquad |\boldsymbol{v}_\perp| = v\sin\alpha \tag{6.3}$$

where α (pitch-angle): $\angle(\boldsymbol{v}, \boldsymbol{B})$

No acceleration parallel to the $\boldsymbol{B}$-field $\Longrightarrow$ only $v_\perp$ is interesting $\Longrightarrow$ circular motion:

$$m_\mathrm{e}a_\perp = \frac{\gamma m_\mathrm{e} v_\perp^2}{R} = \frac{e}{c}v_\perp B \iff \frac{v_\perp}{R} = \frac{eB}{\gamma m_\mathrm{e} c} = \frac{\omega_\mathrm{L}}{\gamma} = \omega_\mathrm{B} \tag{6.4}$$

where $\omega_\mathrm{L} = 2\pi\nu_\mathrm{L}$: Larmor-frequency

(also Cyclotron-frequency, gyro-frequency)

Numerical values

Numerically, Larmor frequency is

$$\nu_\mathrm{L} = 2.8\,\mathrm{MHz}\cdot\frac{B}{1\,\mathrm{G}} \tag{6.5}$$

The radius of the orbit (Larmor radius) is

$$R = \frac{\gamma v_\perp}{\omega_\mathrm{L}} \approx 2\,\mathrm{AU}\cdot\frac{E}{1\,\mathrm{GeV}}\cdot\left(\frac{B}{10^{-6}\,\mathrm{G}}\right)^{-1} \tag{6.6}$$

i.e., small on cosmical scales ($\Longrightarrow$ MHD gets possible).

Radiated Energy

Electrodynamics: Radiation of an accelerated electron:

$$P_{\mathrm{em}} = \frac{2e^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right) \tag{6.7}$$

where a is the acceleration.

(Derivation by Lorentz-transforming the classical Larmor formula, see, e.g., Shu)

In case of circular motion, $a_\perp = \omega_{\mathrm{B}} v_\perp$. Hence

$$P_{\mathrm{em}} = \frac{2e^2}{3c^3}\gamma^4 \frac{v_\perp^2 e^2 B^2}{\gamma^2 m_{\mathrm{e}}^2 c^2} = 2\beta^2\gamma^2 c \cdot \sigma_{\mathrm{T}} \cdot U_{\mathrm{B}} \cdot \sin^2\alpha \tag{6.8}$$

where

- $U_{\mathrm{B}} = B^2/8\pi$ (Energy density of the B-field),
- $\sigma_{\mathrm{T}} = \frac{8\pi e^4}{3m_{\mathrm{e}}^2 c^4}$ (Thomson-cross section).

Presence of σ_{T} due to quantum electrodynamics: Derivation of synchrotron-radiation in frame of reference of electron via interaction of electron with a virtual photon of the magnetic field (i.e., Compton scattering with virtual photon).

Radiated Energy

To obtain total emitted energy: Integrate over all electrons.

Assumption: Isotropic velocity distribution.

Average pitch angle

$$\left\langle \sin^2\alpha \right\rangle = \frac{1}{4\pi}\int_0^{4\pi} \sin^2\alpha \,\mathrm{d}\Omega = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin^2\alpha \sin\alpha \;\mathrm{d}\alpha = \frac{2}{3} \tag{6.9}$$

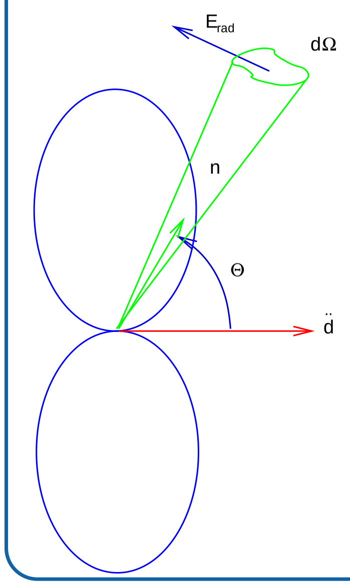
therefore for $\beta \longrightarrow 1$:

$$\langle P_{\mathrm{em}}\rangle = \frac{4}{3}\beta^2\gamma^2 c\sigma_{\mathrm{T}} U_{\mathrm{B}} \sim 1.6\times10^{-2}\,\mathrm{eV\,s^{-1}} \cdot \left(\frac{B^2}{8\pi}\right)\left(\frac{E}{m_{\mathrm{e}}c^2}\right)^2 \tag{6.10}$$

Note: Since $E = \gamma m_{\mathrm{e}} c^2 \Longrightarrow P \propto E^2 U_{\mathrm{B}}$.

Note: $P_{\mathrm{em}} \propto \sigma_{\mathrm{T}} \propto m_{\mathrm{e}}^{-2} \Longrightarrow$ Synchrotron radiation from charged particles with larger mass (protons,...) is negligible.

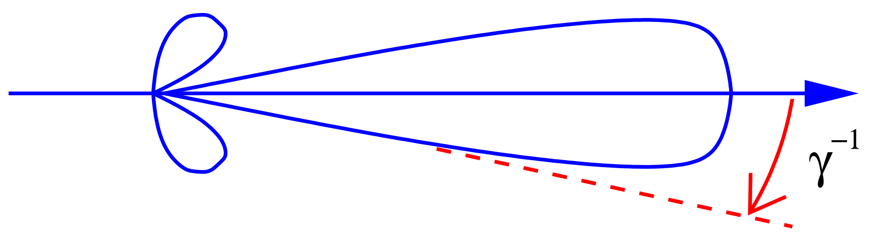
Single Electron spectrum, I



Frame of reference of electron: Emitted radiation has dipole characteristic (see, e.g., Eq. 4.6).

after Rybicki & Lightman, Fig. 3.5

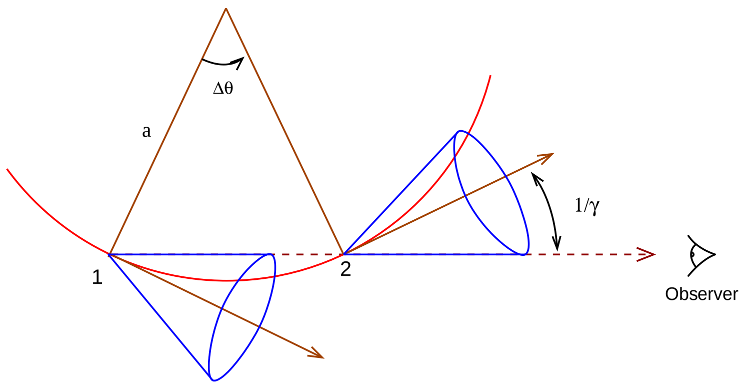
Single Electron spectrum, II



after Rybicki & Lightman, Fig. 4.11d

Lorentz-Transform into laboratory system: Forward Beaming. Opening angle is $\Delta\theta \approx \gamma^{-1}$.

Single Electron spectrum, III

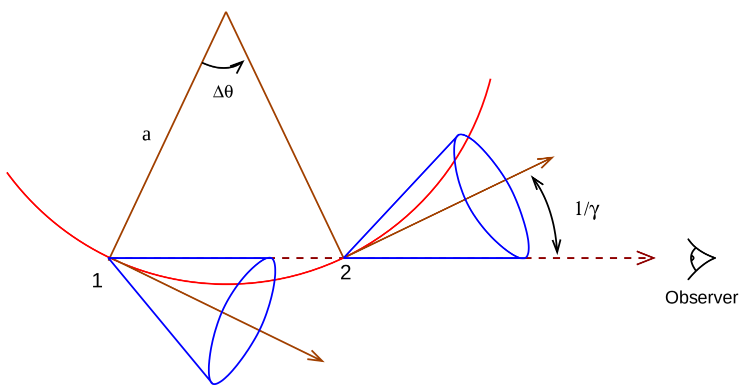


after Rybicki & Lightman, Fig. 6.2

In the electron frame of rest, beam passes observer during time

$$\Delta t = \frac{\Delta\theta}{\omega_\mathrm{B}} = \frac{m_\mathrm{e}c\gamma}{eB}\frac{2}{\gamma} = \frac{2}{\omega_\mathrm{L}} \tag{6.11}$$

Single Electron spectrum, IV



after Rybicki & Lightman, Fig. 6.2

But: Doppler effect shortens duration of pulse (electron is closer to observer at end of beam).

$\Longrightarrow$ Duration of pulse:

$$\tau = \left(1 - \frac{v}{c}\right)\Delta t = (1-\beta)\Delta t \tag{6.12}$$

Single Electron spectrum, V

For $\gamma \gg 1$, i.e., $\beta = v/c \sim 1$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} = (1+\beta)(1-\beta) \sim 2(1-\beta) \tag{6.13}$$

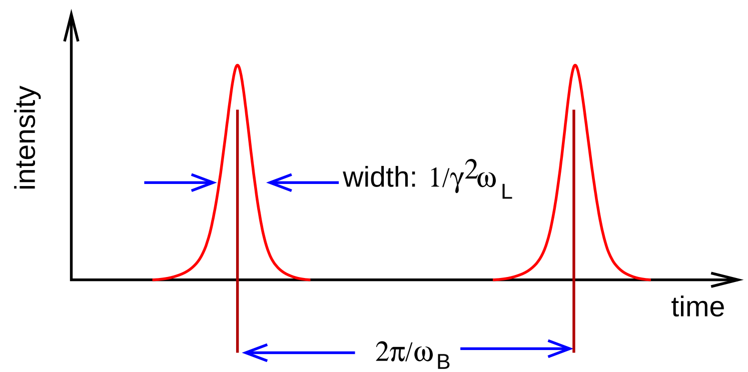
such that

$$\tau = (1-\beta)\Delta t = \frac{\Delta t}{2\gamma^2} = \frac{1}{2\gamma^2}\frac{2}{\omega_\mathrm{L}} = \frac{1}{\gamma^2\omega_\mathrm{L}} \tag{6.14}$$

Thus the characteristic frequency of the radiation is given by

$$\omega_\mathrm{c} = \gamma^2\omega_\mathrm{L} = \frac{eB}{m_\mathrm{e}c}\left(\frac{E}{m_\mathrm{e}c^2}\right)^2 \tag{6.15}$$

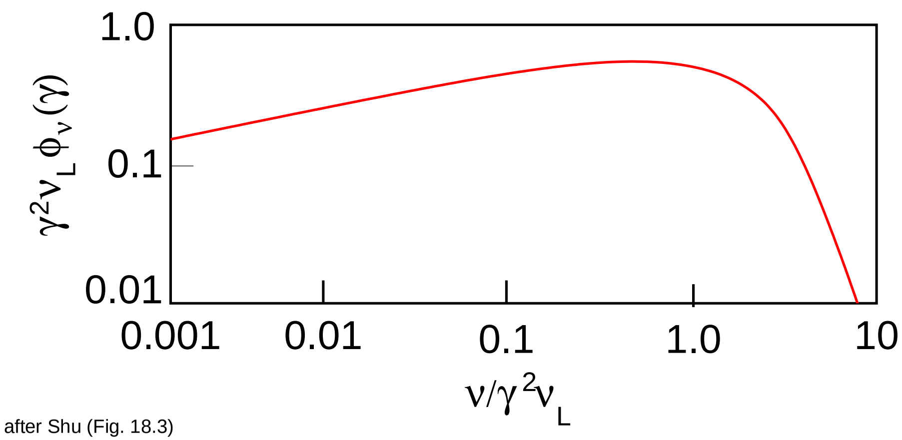
Resulting Field, I



after Shu (Fig. 18.2)

The observed time-dependent E-Field, $E(t)$, from one electron is a sequence of pulses of width τ, separated in time by Δt.

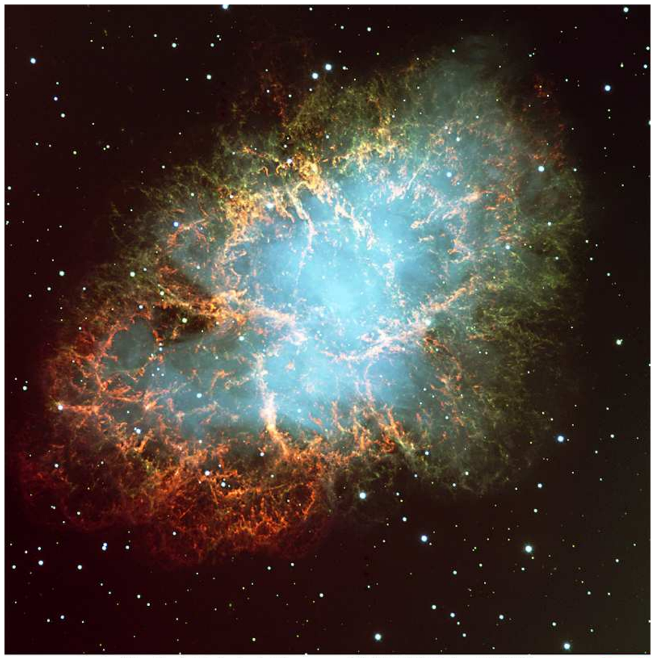
Resulting Field, II



after Shu (Fig. 18.3)

Derive spectrum by Fourier-transforming $E(t)$ (analoguous to bremsstrahlung).

τ small $\Longrightarrow$ relevant frequency range quite large.



The Crab Nebula in Taurus (VLT KUEYEN + FORS2)

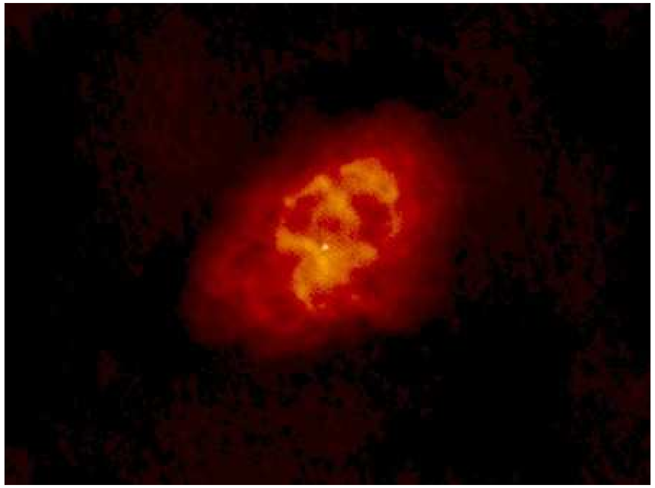
ESO PR Photo 40f/99 (17 November 1999) © European Southern Observatory

The characteristic frequency was

$$\nu_c = \frac{\omega_c}{2\pi} = \\ = 630\,\text{MHz}\left(\frac{B}{10^{-4}\,\text{G}}\right)\cdot \\ \left(\frac{E/m_ec^2}{10^3}\right)^2 \quad (6.15)$$

Optical light has $\nu \sim 10^8$ MHz. To emit this frequency, the electrons must have $\gamma \sim 10^6$ for a typical B-field. These values are typical for the Crab nebula.

Numerical Values



Crab nebula at 90 cm (NRAO 300′ telescope), resolution 1.3″

The Crab nebula is also bright in the radio $\Longrightarrow$ consequence of the broad synchrotron spectrum

Crab nebula is seen up to the X-rays!

Power-law distribution

In many cases, synchrotron radiation is emitted from electrons which have a power-law distribution

$$n(\gamma)\,\mathrm{d}\gamma = n_0\gamma^{-p}\,\mathrm{d}\gamma \quad (6.16)$$

This process is called nonthermal synchrotron radiation

Power law electron distribution results, e.g., from a variety of different acceleration processes.

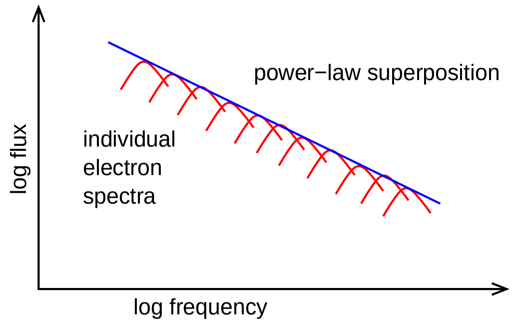
Spectral energy distribution P_ν of an electron with total energy $E = \gamma m_e c^2$ can be written as

$$P_\nu(\gamma) = \frac{4}{3}\beta^2\gamma^2 c\sigma_T U_B \phi_\nu(\gamma) \quad (6.17)$$

where the spectral shape is described by a function $\phi_\nu(\gamma)$ where

$$\int \phi_\nu(\gamma)\,\mathrm{d}\gamma = 1 \quad (6.18)$$

Power-law distribution



after Shu, Fig. 18.4

Assume that photons are only emitted at frequency $\gamma^2\nu_\mathrm{L}$ (good approximation since the spectrum has strong peak there), i.e.,

$$\phi_\nu(\gamma) \sim \delta(\nu - \gamma^2\nu_\mathrm{L}) \quad (6.19)$$

Therefore the emitted power (=spectrum) is

$$P_\nu = \int_1^\infty \langle P_\nu(\gamma)\rangle\, n(\gamma)\, \mathrm{d}\gamma \quad (6.20)$$

Power-law distribution

Therefore

$$P_\nu = \int_1^\infty \frac{4}{3}\beta^2\gamma^2 c\sigma_\mathrm{T} U_\mathrm{B}\delta(\nu - \gamma^2\nu_\mathrm{L}) n_0 \gamma^{-p}\, \mathrm{d}\gamma \quad (6.21)$$

since $\gamma \gg 1$: $\beta \approx 1$

$$= A\int_1^\infty \gamma^{2-p}\delta(\nu - \gamma^2\nu_\mathrm{L})\, \mathrm{d}\gamma \quad (6.22)$$

substituting $\nu' = \gamma^2\nu_\mathrm{L}$, i.e., $\mathrm{d}\nu' = \nu_\mathrm{L} 2\gamma\, \mathrm{d}\gamma$

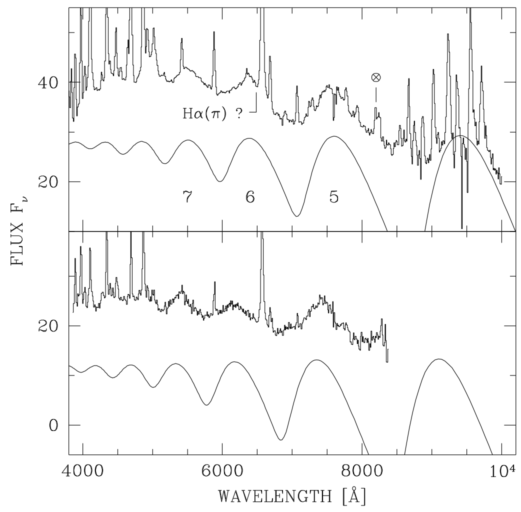
$$= B\int_{\nu_\mathrm{L}}^\infty \gamma^{1-p}\delta(\nu - \nu')\, \mathrm{d}\nu' \quad (6.23)$$

since $\gamma = (\nu'/\nu_\mathrm{L})^{1/2}$, we find

$$P_\nu = \frac{2}{3} c\sigma_\mathrm{T} n_0 \frac{U_\mathrm{B}}{\nu_\mathrm{L}} \left(\frac{\nu}{\nu_\mathrm{L}}\right)^{-\frac{p-1}{2}} \quad (6.24)$$

The spectrum of an electron power-law distribution is a power-law!

Cyclotron emission



For weak magnetic fields and nonrelativistic electron distributions, the different orders of emission can be observed directly in AM Her objects, i.e., strongly magnetized and accreting white dwarfs in a binary system.

Example: The AM Her object VV Pup shows varying cyclotron lines from two magnetic poles with $B = 31$ MG and $B = 54$ MG (seen over the rotation of the white dwarf).

Schwope & Beuermann, 1997, AN 318, 111

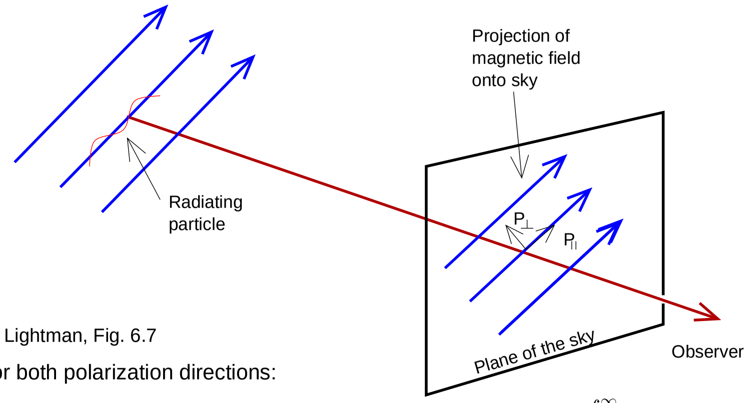
Summary

What we have done so far:

1. Motion of the electron
2. Radiation characteristic from relativistic motion
3. Doppler-effect
4. Integration over electron distribution

It is possible to do the same analytically without any approximations. This is too complicated to be done here. See the references for details.

Result of exact calculation



Rybicki & Lightman, Fig. 6.7

Result for both polarization directions:

$$\begin{pmatrix} P_\parallel \\ P_\perp \end{pmatrix} = \frac{\sqrt{3}e^3B}{2\ mc^2}\begin{pmatrix} F(\nu/\nu_c) - G(\nu/\nu_c) \\ F(\nu/\nu_c) + G(\nu/\nu_c) \end{pmatrix} \quad \text{where} \quad \begin{array}{l} F(x) = x\int_x^\infty K_{5/3}(y)\,\mathrm{d}y \\ G(x) = xK_{2/3}(x) \end{array} \tag{6.25}$$

and K_i are modified Bessel-functions of ith order

Polarization allows to measure the magnetic field direction

$$F(x) = x\int_x^\infty K_{5/3}(\eta)d\eta \quad \text{and} \quad G(x) = xK_{2/3}(x)$$

x	$F(x)$	$G(x)$	x	$F(x)$	$G(x)$
0	0	0	0.90	0.694	0.521
0.001	0.213	0.107	1.0	0.655	0.494
0.005	0.358	0.184	1.2	0.566	0.439
0.01	0.445	0.231	1.4	0.486	0.386
0.025	0.583	0.312	1.6	0.414	0.336
0.050	0.702	0.388	1.8	0.354	0.290
0.075	0.772	0.438	2.0	0.301	0.250
0.10	0.818	0.475	2.5	0.200	0.168
0.15	0.874	0.527	3.0	0.130	0.111
0.20	0.904	0.560	3.5	0.0845	0.0726
0.25	0.917	0.582	4.0	0.0541	0.0470
0.29	0.918	0.592	4.5	0.0339	0.0298
0.30	0.918	0.596	5.0	0.0214	0.0192
0.40	0.901	0.607	6.0	0.0085	0.0077
0.50	0.872	0.603	7.0	0.0033	0.0031
0.60	0.832	0.590	8.0	0.0013	0.0012
0.70	0.788	0.570	9.0	0.00050	0.00047
0.80	0.742	0.547	10.0	0.00019	0.00018

Total Spectrum

The total emitted power for monoenergetic electrons is

$$P(\nu) = P_\parallel(\nu) + P_\perp(\nu) \propto F(\nu) \tag{6.26}$$

As before, the total emitted spectrum is found by integrating over the electron energy distribution. For a power-law:

$$\begin{pmatrix} P_\parallel(\nu) \\ P_\perp(\nu) \end{pmatrix} = \left(\frac{\sqrt{3}}{2}\right) n_0 \frac{e^3B}{m_\mathrm{e}c^2}\begin{pmatrix} J_F - J_G \\ J_F + J_G \end{pmatrix}\left(\frac{2\nu}{3\nu_\mathrm{L}}\right)^{-(p-1)/2} \tag{6.27}$$

where

$$J_F = \frac{2^{(p+1)/2}}{p+1}\Gamma\left(\frac{p}{4}+\frac{19}{12}\right)\Gamma\left(\frac{p}{4}-\frac{19}{12}\right) \tag{6.28}$$

$$J_G = 2^{(p-3)/2}\Gamma\left(\frac{p}{4}+\frac{7}{12}\right)\Gamma\left(\frac{p}{4}-\frac{1}{12}\right) \tag{6.29}$$

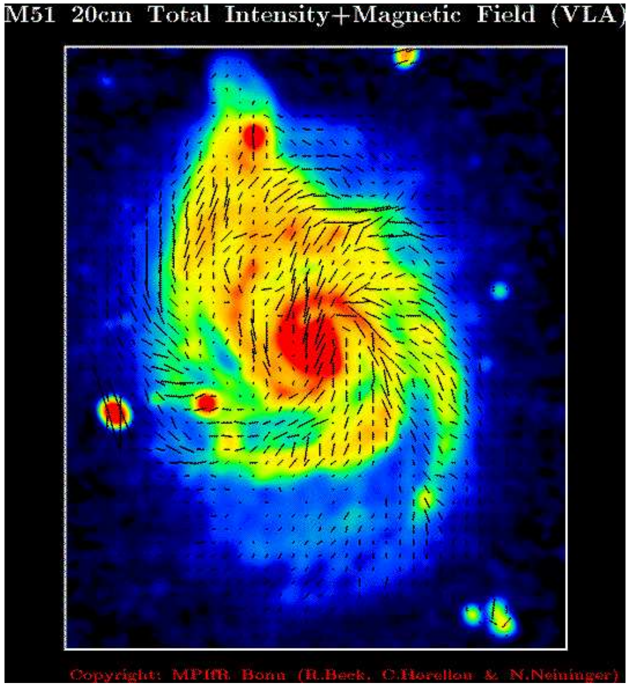
Degree of Polarization

The degree of polarization is defined by

$$\frac{P_\perp - P_\parallel}{P_\perp + P_\parallel} = \frac{J_G}{J_F} = \frac{p+1}{p+7/3} \tag{6.30}$$

For $p = 2.5$ the degree of polarization is $\sim 70\%$. This is very large!!

Caveat: Faraday-rotation and B-field inhomogeneities can decrease the degree of polarization



B-field vectors inferred from the degree of polarization in spiral galaxy M51 by rotation of the observed E-field-vectors by 90°

(Neininger 1992, A&A 263, 30)

Synchrotron Self-Absorption, I

Approach similar to approach used to derive bremsstrahlung self-absorption and thus the result is quantitatively similar:

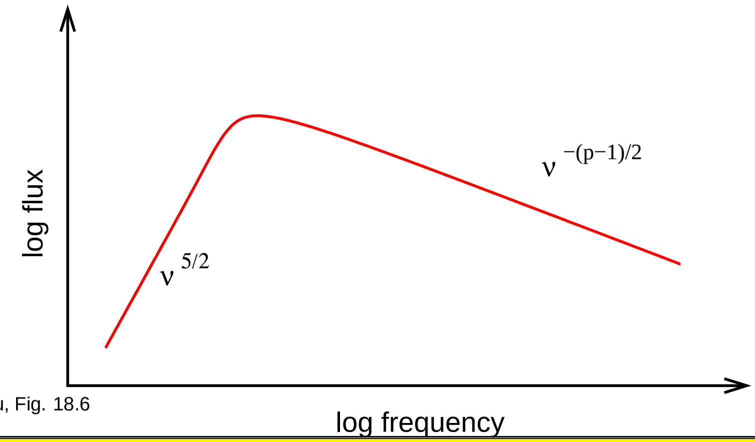
Result: Below a cut-off frequency, the electrons are optically thick for the synchrotron radiation: Synchrotron Self-Absorption.

For an electron power law distribution the optically thick spectrum is

$$P_\nu \propto B^{-1/2}\nu^{5/2} \tag{6.31}$$

(independent of the electron energy index p!).

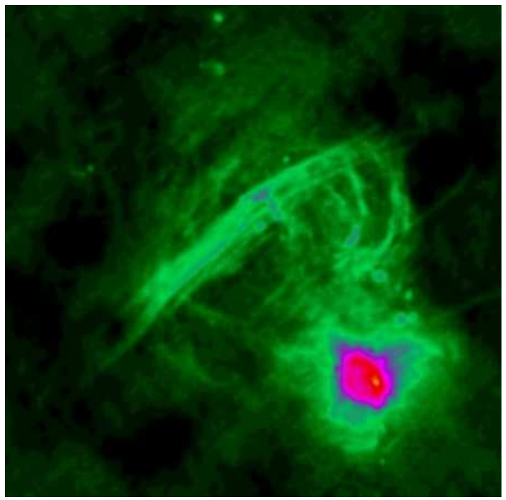
Synchrotron Self-Absorption, II



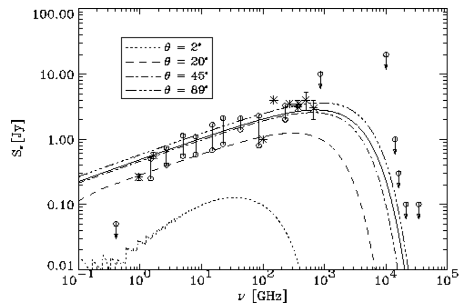
after Shu, Fig. 18.6

For low frequencies: $P_\nu \propto \nu^{5/2}$, For large frequencies: $P_\nu \propto \nu^{-(p-1)/2}$.

- At very high frequencies, additional break due to electron energy losses.
- The transition frequency can be used to measure the strength of the B-Field.

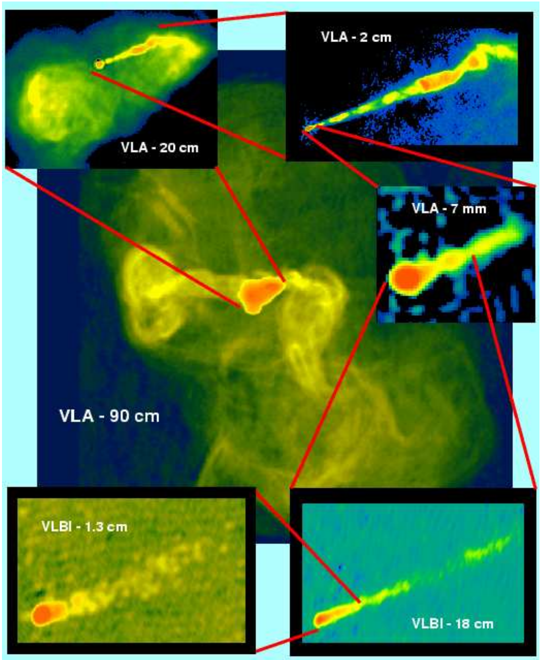


Galactic Center, courtesy NRAO

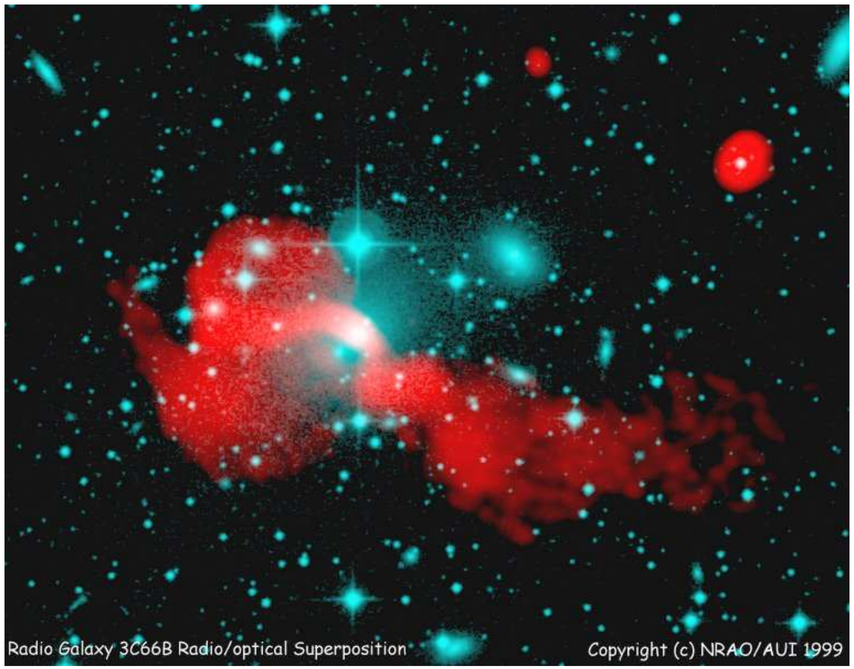


Emission from Sgr A, the galactic center: spectrum characteristic for synchrotron radiation.

Note how emissivity follows B-field structure!



AGN M87; courtesy Frazer Owen



3C66B, courtesy A. Bridle/NRAO