Astrophysical Radiation Processes
Sommersemester 2008
Worksheet 1

Question 1: *Poynting Vector and Flux*

For an electromagnetic wave,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{a}_1 E_0 \mathrm{e}^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \quad \text{and} \quad \mathbf{B}(\mathbf{r},t) = \mathbf{a}_2 B_0 \mathrm{e}^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \tag{w1.1}$$

where the unit vectors $\mathbf{a}_1 \perp \mathbf{a}_2$ and where the unit vector $\mathbf{k} = \mathbf{a}_1 \times \mathbf{a}_2$ points into the direction of propagation of the wave.

a) The energy transported by the wave is given by the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B}) \tag{2.33}$$

Show that for the wave defined above, $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{c}{4\pi} E_0 B_0 \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r})\,\mathbf{k} \tag{w1.2}$$

since only the real part of $\mathbf{E}$ and $\mathbf{B}$ has a physical interpretation.
Remember $e^{i\varphi} = \cos\varphi + i\sin\varphi$.

b) Since the wave has a high frequency, the instantaneous value of $\mathbf{S}$ can typically not be measured. Instead instruments measure the *time-averaged* value of $\mathbf{S}$, which we call the "flux". Show that this time average is given by

$$\langle S\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} S\,\mathrm{d}t = \frac{c}{8\pi} E_0 B_0 = c\frac{B_0^2}{8\pi} \tag{s1.1}$$

For simplicity, you can assume $\mathbf{k}\cdot\mathbf{r} = 0$.
Note that $\int \cos^2 x\,\mathrm{d}x = \frac{1}{2}x + \frac{1}{4}\sin 2x$ and remember that $E_0 = B_0$.

Question 2: *Radiation Pressure*

The following discussion is based on Section 2.7.1 of Imke de Pater and Jack J. Lissauer, 2001, Planetary Sciences, Cambridge: Cambridge Univ. Press.

a) As shown in the lecture, $B^2/8\pi$ has the units of an energy density. Convince yourself that the units of an energy density are the same as that of a pressure. Since we work in cgs units, it is good to know that the unit of force is called a "dyne" ($1\,\mathrm{dyne} = 1\,\mathrm{g\,cm\,s^{-2}}$), while that of energy is an "erg" ($1\,\mathrm{erg} = 1\,\mathrm{g\,cm^2\,s^{-2}}$).

b) Convince yourself that the force excerted onto an area A if the impinging radiation is fully absorbed is given by

$$F_{\mathrm{rad}} = \frac{S}{c}A \tag{s2.1}$$

This quantity is called the *radiation pressure*, it acts in the direction of $\mathbf{S}$.

c) On small particles in the solar system, the radiation pressure from the Sun can be significant. Show that the radiation force onto a particle of area A is

$$\mathbf{F}_{\mathrm{rad}} = \frac{L}{4\pi r^2}\frac{A}{c}Q_{\mathrm{pr}} \tag{s2.1}$$

where Q_{pr} is a correction factor called the *radiation pressure coefficient*.

d) For a spherical particle with density ρ and radius R, show that the ratio between the radiation force and the gravitational force excerted by the Sun is given by

$$\beta = \frac{|\mathbf{F}_{\rm rad}|}{|\mathbf{F}_{\rm grav}|} = \frac{3L}{16\pi cGM} \cdot \frac{Q_{\rm pr}}{R\rho} = 5.7 \times 10^{-5} \frac{Q_{\rm pr}}{\rho R} \propto \frac{Q_{\rm pr}}{\rho R} \tag{s2.1}$$

where the numerical value assumes that R and ρ are measured in cgs units. What is the consequence of $\beta > 1$?

Question 3: *Deriving the Formal Solution to the Equation of Radiative Transfer*

a) By multiplying both sides of the equation of radiative transfer

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu = S_\nu - I_\nu \tag{2.92}$$

with e^{τ_ν} and some simple algebra, show that

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu} \left(\mathrm{e}^{\tau_\nu} I_\nu\right) = \mathrm{e}^{\tau_\nu} S_\nu \tag{w3.1}$$

b) Show by separation of variables that Eq. (w3.1) can be written as

$$\mathrm{e}^{\tau_\nu} I_\nu(\tau_\nu) - I_\nu(0) = \int_0^{\tau_\nu} \mathrm{e}^{\tau'_\nu} S_\nu(\tau'_\nu) \mathrm{d}\tau'_\nu \tag{s3.1}$$

and that therefore

$$I_\nu(\tau_\nu) = I_\nu(0)\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} \mathrm{e}^{-(\tau_\nu - \tau'_\nu)} S(\tau'_\nu) \mathrm{d}\tau'_\nu \tag{2.93}$$

Question 4: *The Formal Solutions of the Equation of Radiative Transfer*

a) Show that for a homogeneous medium, i.e., a medium in which the source function $S_\nu = j_\nu/\alpha_\nu$ is independent of place, the formal solution of the transfer equation 2.93 is given by

$$I_\nu = I_{\nu,0}\mathrm{e}^{-\tau_\nu} + \frac{j_\nu}{\alpha_\nu}\left(1 - \mathrm{e}^{-\tau_\nu}\right) \tag{w4.1}$$

b) To first order, we can approximate stellar atmospheres by semi-infinite plane-parallel slabs of gas. For many stellar atmospheres, it is a good assumption that the source function is given by

$$S(\tau) = a + b\tau \tag{s4.1}$$

where a and b are constants and where τ is the optical depth *into* the atmosphere, i.e., opposite to the direction of light propagation (so far, we've always measured τ along the direction of propagation of light). For this reason the equation of transfer is

$$\cos\theta \frac{\mathrm{d}I(\theta)}{\mathrm{d}\tau} = I(\theta) - S \tag{s4.2}$$

We will now calculate the emergent intensity from the stellar atmosphere as a function of angle θ from the normal. The calculation will be somewhat less messy if we make use of a rarely employed function called the "secant", defined by

$$\sec\theta = \frac{1}{\cos\theta} \tag{s4.3}$$

Apart from that the calculation will be rather similar to the derivation of the formal solution of the equation of transfer:

i. First, multiply the transfer equation with $\mathrm{e}^{-\tau\sec\theta}$ and show that

$$\frac{\mathrm{d}\left(I\mathrm{e}^{-\tau\sec\theta}\right)}{\mathrm{d}\left(\tau\sec\theta\right)} = -S\mathrm{e}^{-\tau\sec\theta} \tag{s4.4}$$

where one has to remember the chain rule and

$$\frac{1}{\sec\theta}\frac{\mathrm{d}I(\theta)}{\mathrm{d}\tau} = \frac{\mathrm{d}I(\theta)}{\mathrm{d}\left(\tau\sec\theta\right)} \tag{s4.5}$$

ii. To derive the intensity $I(0,\theta)$ at the surface, integrate Eq. (w4.11) over $\mathrm{d}(\tau\sec\theta)$ from 0 to ∞ to show that

$$I(0,\theta) = \int_0^\infty S(\tau)\mathrm{e}^{-\tau\sec\theta}\mathrm{d}(\tau\sec\theta) \tag{s4.1}$$

What is the interpretation of this equation?

iii. Now insert $S(\tau)$ from Eq. (w4.8) to derive

$$I(0,\theta) = a + b\cos\theta = S(\tau = \cos\theta) \tag{s4.1}$$

that is, for all angles we see an emerging intensity equal to the value of the source function at $\tau = 1$ along the line of sight. A consequence of this result is the so-called *limb-darkening* of the Sun.

Remember $\int x\mathrm{e}^{-x}\,\mathrm{d}x = -\mathrm{e}^{-x}(x+1)$.